(Scalar) function: number x → $\boxed{f}$ → number $f(x)$ e.g. $f(x) = x^2+1$

$\underset{2}{x} \to \boxed{} \to \overset{f(x)}{5}$

Functional I

function $f(x)$ → $\boxed{I}$ → number I

x^2+1 → 12

x^3-5 → $9\frac{1}{2}$

$$I = \int_{x=x_1}^{x_2} G\left[f(x), \frac{df(x)}{dx} ; x\right] dx$$

explicit x dependence not in $f(x)$, $f'(x)$, etc

possibly more $\frac{d^2f}{dx^2}$, $\frac{d^3f}{dx^3}$, etc.

x is the variable

f is the function input to the functional I.

G is the function of the function f and its derivatives.

e.g.

$f(x) = x^2+1$ $\quad f'(x) = 2x \quad G = \frac{f^3}{f'} + \sin(f)\, x$ explicit dependence

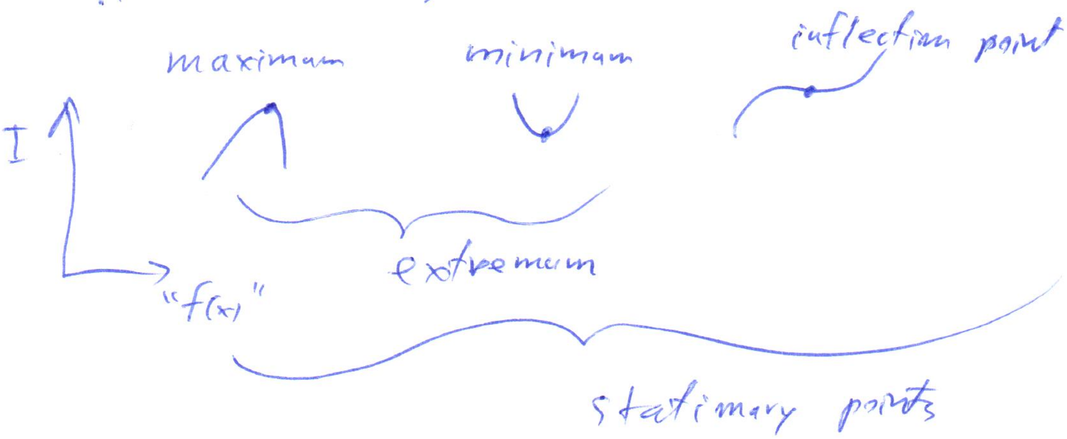
$$= \frac{(x^2+1)^3}{2x} + \sin(x^2+1)\, x$$

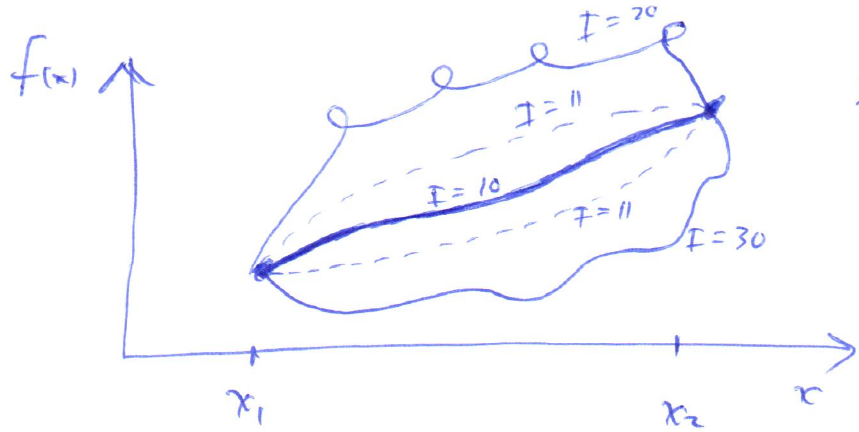
$$I = \int_{x=x_1}^{x_2} \left[\frac{(x^2+1)^3}{2x} + \sin(x^2+1)\, x\right] dx = \text{some number}$$

e.g. $x_1 = 1$, $x_2 = 2$, $I = 13.1217\ldots$

We look for functions $f(x)$ that make I stationary:

first order changes in $f(x)$ do not result in a change in I.





string on table - push around.

call the function that makes I stationary (minimum here) $f_0(x)$.

$$f_\alpha(x) \equiv f_0(x) + \alpha \eta(x)$$ where $\eta(x_1) = 0 = \eta(x_2)$

endpoints fixed

$$f_\alpha'(x) = f'(x) + \alpha \eta'(x)$$

---

want $\delta I = 0 = \dfrac{\partial I}{\partial \alpha} d\alpha$

$$I = \int_{x=x_1}^{x_2} G\left[f(x), f'(x); x\right] dx$$

$$\delta I = \int_{x=x_1}^{x_2} \left[\frac{\partial G}{\partial f} \underbrace{\frac{df}{d\alpha}}_{\eta(x)} d\alpha + \frac{\partial G}{\partial f'} \underbrace{\frac{df'}{d\alpha}}_{\eta'(x)} d\alpha\right] dx$$

$$\delta I = \int_{x=x_1}^{x_2} \left[\frac{\partial G}{\partial f} \eta + \frac{\partial G}{\partial f'} \eta'\right] d\alpha\, dx$$

use integration by parts to move the derivative off the function $\eta(x)$. Then both terms will be proportional to $\eta(x)$.

$$\int_{x=x_1}^{x_2} \frac{\partial G}{\partial f'} \frac{d\eta}{dx} d\alpha\, dx = \frac{\partial G}{\partial f'} \eta(x) \Big|_{x=x_1}^{x_2} - \int_{x=x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial G}{\partial f'}\right) \eta(x)\, d\alpha\, dx$$

"surface term" is zero since $\eta(x_1) = 0 = \eta(x_2)$

$$\delta I = \int_{x=x_1}^{x_2} \left[\frac{\partial G}{\partial f}\eta - \frac{d}{dx}\left(\frac{\partial G}{\partial f'}\right)\eta \right] d\alpha\, dx$$

$$\delta I = \int_{x=x_1}^{x_2} \left[\frac{\partial G}{\partial f} - \frac{d}{dx}\left(\frac{\partial G}{\partial f'}\right) \right] \eta(x)\, d\alpha\, dx = 0 \qquad \forall\, \eta(x)$$

arbitrary

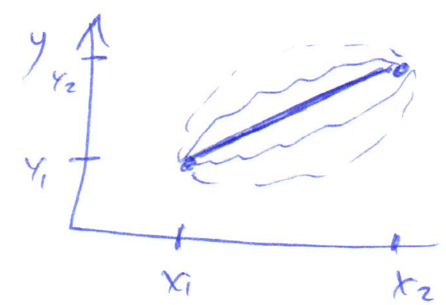
$$\Longrightarrow \frac{\partial G}{\partial f} - \frac{d}{dx}\left(\frac{\partial G}{\partial f'}\right) = 0$$

called Euler-Lagrange Equation

This will generate a differential equation involving $f(x)$, $f'(x)$, $f''(x)$, etc. which can then be solved for $f(x)$ which makes I stationary.

Examples

① Shortest distance between two points in the plane (xy-plane)



the length of the curve l (a number) will be the functional. We look for curves (paths) $y(x)$ that minimize l.

arc length $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$= \sqrt{1 + y'^2}\, dx$

$$l = \int_{x=x_1}^{x_2} ds = \int_{x=x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

so $G = \sqrt{1 + y'^2}$

in general $G[y(x), y'(x); x]$

none none

$$\delta l = 0 \Longrightarrow \frac{\partial G}{\partial y} - \frac{d}{dx}\frac{\partial G}{\partial y'} = 0$$

$$\frac{\partial G}{\partial y} = 0 \qquad \frac{\partial G}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

Euler-Lagrange: $0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \longrightarrow \frac{y'}{\sqrt{1+y'^2}} = c$ (constant)

Solve for $y'(x)$: $y'(x) = \frac{c}{\sqrt{1-c^2}} \equiv a$ (another constant)

$y'(x) = \frac{dy(x)}{dx} = a \Rightarrow y(x) = \underset{\text{slope}}{a}x + \underset{\text{y intercept}}{b}$ – straight line

choose $\underline{a}$ and $\underline{b}$ so that $y(x)$ goes through the fixed endpoints (x_1, y_1) + (x_2, y_2)

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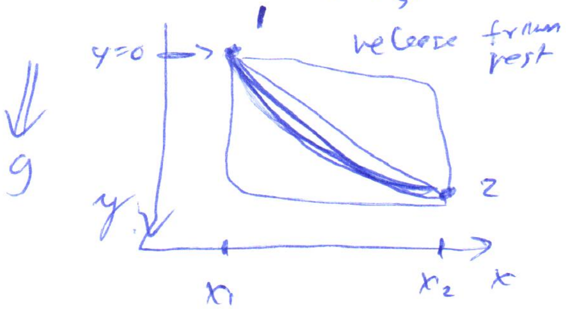
(2) Brachistochrone "shortest time"



In a constant gravitational field, find the path $y(x)$ that minimizes the time from (x_1, y_1) to (x_2, y_2)

functional = ? = t

$$t = \int_1^2 \frac{ds}{v} \qquad \delta t = 0$$

Get the speed v from energy conservation $\frac{1}{2}mv^2 = mgy$

$\Rightarrow v = \sqrt{2gy}$ $\qquad ds = \sqrt{dx^2 + dy^2} = \sqrt{1+y'^2}\, dx$

$$t = \int_{x=x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx \quad ; \quad G[y(x), y'(x); x] = ? = \sqrt{\frac{1+y'^2}{2gy}}$$

$$\frac{\partial G}{\partial y} = \sqrt{\frac{1+y'^2}{2g}} \left(-\frac{1}{2}\right) y^{-3/2} \quad \frac{\partial G}{\partial y'} = \frac{1}{\sqrt{2gy}} \frac{y'}{\sqrt{1+y'^2}}$$

E-L: $\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0$

$$\Rightarrow -\frac{1}{2}\sqrt{\frac{1+y'^2}{2gy^3}} - \frac{d}{dx}\left[\frac{1}{\sqrt{2gy}} \frac{y'}{\sqrt{1+y'^2}}\right] = 0$$

complicated differential equation

2nd order – y''

not clear if linear in $y(x)$

Two ways to fix this: A) Beltrami form of E-L equation.

B) change $x \leftrightarrow y$.

FIRST form:

$$\frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0$$

Total derivative of G with respect to x

$$\frac{dG}{dx} = \underset{\substack{\uparrow \\ \text{explicit} \\ x \text{ dependence}}}{\frac{\partial G}{\partial x}} + \frac{\partial G}{\partial y}\underbrace{\frac{dy}{dx}}_{y'} + \frac{\partial G}{\partial y'}\underbrace{\frac{dy'}{dx}}_{y'' = \frac{d^2y}{dx^2}}$$

$$\frac{dG}{dx} = \frac{\partial G}{\partial x} + \underbrace{\frac{\partial G}{\partial y} y'}_{\text{solve for this}} + \frac{\partial G}{\partial y'} y''$$

$$\frac{\partial G}{\partial y} y' = \frac{dG}{dx} - \frac{\partial G}{\partial x} - \frac{\partial G}{\partial y'} y''$$

$$\frac{\partial G}{\partial y} y' = \left[\frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right] y'$$ from 1st form, Subtract.

$$0 = \underline{\frac{dG}{dx}} - \frac{\partial G}{\partial x} - \underline{\frac{\partial G}{\partial y'} y''} - \underline{\left[\frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right)\right] y'}$$

$$\boxed{0 = -\frac{\partial G}{\partial x} + \frac{d}{dx}\left[G - \frac{\partial G}{\partial y'} y'\right]}$$

Beltrami form (2nd form) of Euler-Lagrange

Looks more complicated...

... but, if $\frac{\partial G}{\partial x} = 0$ (G has no explicit x dependence, like our examples) then

$$\frac{d}{dx}\left[G - \frac{\partial G}{\partial y'} y'\right] = 0 \implies \underbrace{G - \frac{\partial G}{\partial y'} y' = \text{constant}}_{\Rightarrow\ 1^{st} \text{ order D.E.}}$$

Back to the Brachistochrone!