$100 – physics check to Olness by Thursday (tomorrow)

167

Separation of Variables

① Cartesian $\qquad \nabla^2 \Phi = 0$

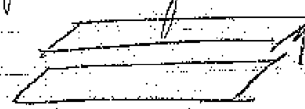
ⓐ One dimension

$$0 = \nabla^2 \Phi(\vec{r}) = \vec{\nabla} \cdot [\vec{\nabla}\Phi(\vec{r})] = \text{div}[\text{grad}\,\Phi(\vec{r})]$$

tensor $\quad = g^{ij}\Phi_{,ij}$

in one dim: $\dfrac{d^2\Phi(x)}{dx^2} = 0$ (Laplace's Eqn) homogeneous version; Poisson's is the non homogeneous version → charge between the plates

example: 2 infinite capacitor plates

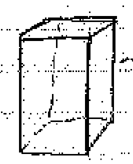


↑ x distance btwn plates

$$\frac{d^2\Phi(x)}{dx^2} = 0 \qquad \Phi(x) = Ax + B \quad \text{(just integrate twice)}$$

Two Dimensions:

infinite in z-direction



$$\nabla^2\Phi(x,y) = 0$$

Assume:

$\Phi(x,y) = f(x)\,g(y)$ Separation of variables

* there's a uniqueness theorem, so any answer you get is the *only* answer ☺

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[f(x)\,g(y)\right] = 0$$

$$g(y)\frac{\partial^2 f(x)}{\partial x^2} + f(x)\frac{\partial^2 g(y)}{\partial y^2} = 0 \quad \leftarrow \text{other terms are } 0$$

$$g(y)\,f''(x) + f(x)\,g''(y) = 0 \quad \leftarrow \text{dividing by } \Phi$$

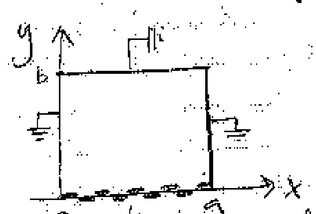
$$\frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} = 0$$

goal: find voltage w/in plates or in pipe knowing only the voltages around channel or on plates

must cancel each other for any value of f(x) or g(y) ∴ so they must be constants

$$\frac{f''(x)}{f(x)} = -\alpha^2 \qquad \frac{g''(y)}{g(y)} = \alpha^2$$

Use superposition (can always use when eqn is linear) ground all but one at a time & then add them all up at the end



looking down the pipe (z direction is useless)

arbitrary function V(x)

boundary condition ⇒ must vanish @ at least 2 places along x (when $x=0$ & $x=a$)

i) need a complete set of functions of x.

ii) need x functions to vanish in at least 2 places ~ sines & cosines (exp won't cut it ⇒ therefore make $\frac{f''(x)}{f(x)} = -\alpha^2$ so you get sin/cos solutions

$$f''(x) + \alpha^2 f(x) = 0 \Rightarrow f(x) = A\cos(\alpha x) + B\sin(\alpha x)$$

several ways to write this

$$g''(y) - \alpha^2 g(y) = 0 \Rightarrow g(y) = Ce^{\alpha y} + De^{-\alpha y}$$

cosh & sinh

Boundary Conditions:

$\Phi(0,y)=0$ (left) | $\Phi(x,b)=0$ (top)

$\Phi(a,y)=0$ (right) | $\Phi(x,0)=V(x)$ (bottom)

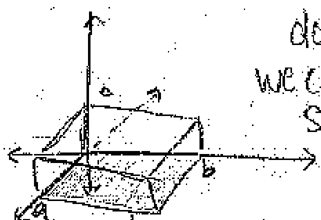
↓ tells you that $\alpha=\frac{n\pi}{a}$ quantization

↓ find the Fourier Coefficients (already done previously)

And now in 3D: still Cartesian, where $\Phi(x,y,z)=f(x)g(y)h(z)$

$$\nabla^2\Phi(\vec{r})=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)f(x)g(y)h(z)$$

$$\frac{f''(x)}{f(x)}+\frac{g''(y)}{g(y)}+\frac{h''(z)}{h(z)}=0$$



don't have to have stripes of voltages → we can have patches & other arbitrary shapes → use superposition again to find total solution

apply voltage to bottom $=V(x,y)$ again, arbitrary & ground 5 other faces

2 separation constants, α^2 & β^2

$$\frac{f''(x)}{f(x)}=-\alpha^2 \quad \frac{g''(y)}{g(y)}=-\beta^2, \text{ then } \frac{h''(z)}{h(z)}=\alpha^2+\beta^2$$

To choose signs of separation constants, use same logic as in 2D for this bottom face, you need sines & cosines for x & y, so their separation constants are negative, forcing z's constant to be neg ~ exp.

$f(x) = A\cos(\alpha x) + B\sin(\alpha x)$
$g(y) = C\cos(\beta y) + D\sin(\beta y)$
$h(z) = Ee^{\sqrt{\alpha^2+\beta^2}z} + Fe^{\sqrt{\alpha^2+\beta^2}z}$

Boundary Conditions

$\Phi(0,y,z) = 0$ (back face) ~ grounded

$\Rightarrow f(0) = 0 = A\cos(\alpha \cdot 0) + B\sin(\alpha \cdot 0) \therefore A = 0$

Similarly, $\Phi(x,0,z) = 0$ (left side)
$g(0) = 0 = C\cos(\beta \cdot 0) + D\sin(\beta \cdot 0) \quad \therefore C = 0$
$0 = C \cdot 1 + 0$

front face $\Phi(a,y,z) = 0$ front face

$f(a) = 0 = 0 \cdot \cos(\alpha \cdot a) + B\sin(\alpha a)$
$B \neq 0$ b/c it would make the entire function 0
$\therefore \sin \alpha a = 0 \quad \alpha a = n\pi \quad \alpha = \frac{n\pi}{a} \quad n = 1, 2, 3 \ldots$

$\Phi(x,b,z) = 0$ (right side)
$\beta = \frac{m\pi}{b} \quad m = 1, 2, 3 \ldots$

n & m are independent & they can't equal zero

(top face) $\Phi(x,y,c) = 0$

$h(c) = 0 = Ee^{+\sqrt{\alpha^2+\beta^2}c} + Fe^{-\sqrt{\alpha^2+\beta^2}c}$
*always use the positive square root

Solve for E in terms of F or F in terms of E

$E = -Fe^{-2\sqrt{\alpha^2+\beta^2}\,c}$ ~ took E to the other side & divided the exp.

$\Phi(x,y,z) = f(x)g(y)h(z)$

$$\Phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \underbrace{BDF}_{G_{nm}} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left[\left(-e^{-2\sqrt{\alpha^2+\beta^2}\,c}\right)\left(e^{\sqrt{\alpha^2+\beta^2}\,z}\right) + e^{-\sqrt{\alpha^2+\beta^2}\,z}\right]$$

really only 1 constant — from boundary condition

* but n & m make it doubly infinite solutions

One last boundary condition to satisfy

$$\Phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} G_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) *$$
$$* \left[\underbrace{\left(-e^{-2\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,c}\right)}_{E} e^{\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,z} + e^{-\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,z}\right]$$

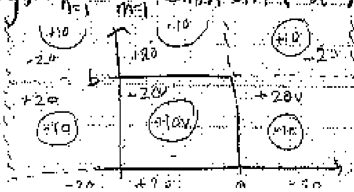
$\Phi(a,b,c) = 0$ → true, but not as powerful a condition as
$\Phi(a,y,z) = 0$

Last (6th) Boundary Condition

$\Phi(x,y,0)$ ~ bottom $= V(x,y)$ (arbitrary)

So go back to equation and plug in 0 for z

$$V(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} G_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) * \left[1 - e^{-2\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,c}\right]$$



make this an odd function

"tile" the plane this way

6:48AM Sunday

Use Fourier Trick:
take dot product w/ sin...

$$\int_{x=0}^{2b}\int^{2a} \sin\left(\frac{m'\pi y}{b}\right)\sin\left(\frac{n'\pi x}{a}\right) V(x,y)\,dx\,dy$$

picks out the n' & m'

$$=\frac{1}{ab}\int_{y=0}^{2b}\int_{x=0}^{2a} \sin\left(\frac{m'\pi y}{b}\right)\sin\left(\frac{n'\pi x}{a}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} G_{nm} \, * \, \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left[1-e^{-2\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,c}\right]dx\,dy$$

$\delta_{nn'} = 1$ if $n=n'$, 0 otherwise $\delta_{mm'} = 1$ if $m=m'$, 0 otherwise

Simplify:

$$G_{n'm'}\left[1-e^{-2\pi c\sqrt{\left(\frac{n'}{a}\right)^2+\left(\frac{m'}{b}\right)^2}}\right]$$

constant, no sum

* now you can remove primes b/c they're just variables ☺

$$G_{nm}=\frac{\frac{1}{ab}\int_{y=0}^{2b}\int_{x=0}^{2a}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)V(x,y)\,dx\,dy}{1-e^{-2\pi c\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}}}$$

$m, n = \{1, 2, 3, \ldots\}$

$G_{11} =$
$G_{12} =$
$G_{21} =$

Cylindrical Symmetry

Boundary Conditions depend only on z

BCs depend only on φ

$$\frac{\partial^2}{\partial \varphi^2}\Phi(\varphi) = 0 \qquad \nabla^2\Phi = 0$$

$$\Downarrow$$

$$\Phi(u) = A\varphi + B$$

BC's depend only on ρ (radial coordinate)

$$\nabla^2\Phi(\vec{r}) = 0$$

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\Phi(\rho)\right] = 0$$

$$\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\Phi(\rho)\right] = 0$$

$$\frac{\left[\rho\frac{\partial}{\partial\rho}\Phi(\rho)\right]}{\rho} = \text{constant} = \frac{C_1}{\rho}$$

$$\frac{d}{d\rho}\Phi(\rho) = \Phi'(\rho) = \frac{C_1}{\rho}$$

$$\underline{\Phi(\rho) = C_1\ln(\rho) + C_2}$$

Cylindrical Polar

BC's dependent on ρ and φ

$\Rightarrow$ Solution does also

Laplace: $\nabla^2 \Phi(\vec{r}) = 0$

$\Phi(\rho, \varphi) = R(\rho) F(\varphi)$

$\uparrow$ not z

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left[\rho \frac{\partial R(\rho)}{\partial \rho}\right] F(\varphi) + \frac{1}{\rho^2} R(\rho) \frac{\partial^2 F(\varphi)}{\partial \varphi^2} = 0$$

divide by $\dfrac{R(\rho)\, F(\varphi)}{\rho^2}$

$$\frac{\rho(\rho R')'}{R(\rho)} + \frac{F''}{F} = 0$$

$$\frac{\rho(\rho R')'}{R} = -\frac{F''}{F} = \alpha^2$$

Case I: $\alpha^2 = 0$

$$\frac{F''}{F} = 0 \qquad \boxed{F = c_1 \varphi + c_2}$$

$$\frac{\rho(\rho R')'}{R} = 0 \qquad \rho(\rho R')' = 0$$

$$(\rho R')' = 0$$

multiply these together to get the whole answer

$$\therefore \rho R' = \text{constant} = d_1$$

$$\int R' = \int \frac{d_1}{\rho} \qquad \boxed{R(\rho) = d_1 \ln \rho + d_2}$$

Case Two: Separation Constant $\alpha^2 > 0$

$$\frac{F''}{F} = -\alpha^2 \Longrightarrow F'' + \alpha^2 F = 0$$

$$\boxed{F = c_3 \cos(\alpha\varphi) + c_4 \sin(\alpha\varphi)}$$

$$\frac{\rho(\rho R')'}{R} = +\alpha^2$$

$$\rho \frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] - \alpha^2 R = 0$$

let $u = \ln(\rho)$ $\quad du = d\ln(\rho) = \frac{1}{\rho} d\rho$

$$\frac{d}{du} = \rho \frac{d}{d\rho}$$

multiply these together to get the full solution

$$\frac{d}{du}\left[\frac{d}{du} R(u)\right] - \alpha^2 R = 0$$

$$R'' - \alpha^2 R = 0$$

$$R(u) = d_3 e^{\alpha u} + d_4 e^{-\alpha u}$$

$$R(\rho) = d_3 e^{\alpha \ln(\rho)} + d_4 e^{-\alpha \ln(\rho)}$$

$$\boxed{R(\rho) = d_3 \rho^{\alpha} + d_4 \rho^{-\alpha}}$$

Case #3

Separation Constant is $-\alpha^2 < 0$

$$\frac{F''}{F} = \alpha^2 \rightarrow F''(\varphi) - \alpha^2 F(\varphi) = 0$$

$$F(\varphi) = c_5 \cosh(\alpha\varphi) + c_6 \sinh(\alpha\varphi)$$ same

$$F(\varphi) = c_7 e^{\alpha\varphi} + c_8 e^{-\alpha\varphi}$$

$$\frac{\rho\,(\rho R')'}{R} = -\alpha^2 \Rightarrow$$

$$R(\rho) = d_5\,\rho^{i\alpha} + d_6\,\rho^{-i\alpha}$$

* throw in an i to change the sign for α - b/c it's squared

$$\sinh(x) \equiv \frac{e^{x} - e^{-x}}{2} \qquad \cosh(x) \equiv \frac{e^{x} + e^{-x}}{2}$$

$$\sin(x) \equiv \frac{e^{ix} - e^{-ix}}{2i} \qquad \cos(x) \equiv \frac{e^{ix} + e^{-ix}}{2}$$

Answer:

$$\Phi(\rho, \varphi) = R(\rho)\,F(\varphi)$$

$$= [c_1\varphi + c_2][d_1 \ln\rho + d_2] \quad \text{Case One}$$

$$+ [c_3 \cos\alpha\varphi + c_4 \sin\alpha\varphi][d_3\rho^{\alpha} + d_4\rho^{-\alpha}] \quad \text{Case Two}$$

$$+ [c_5 \cosh(\alpha\varphi) + c_6 \sinh(\alpha\varphi)][d_5\rho^{i\alpha} + d_6\rho^{-i\alpha}] \quad \text{Case Three}$$

* if you didn't specify if α was real or imaginary, you only need either cases 2 or 3 not both

complete in ρ

$0_{2\pi}$

30^{v}

could also be rewritten:

$$d_7 \cos(\alpha \ln\rho) + d_8 \sin(\alpha \ln\rho)$$

If there is a full cylinder (not Δ)

$0 \le \varphi \le 2\pi$

must $c_1 = 0$ b/c it's linearly dep. on φ

[illegible] must be periodic

(2) If the axis $\rho = 0$ is included

$d_1 = 0, \ d_4 = 0$

$\ln(\rho) \to -\infty \qquad \frac{1}{\rho^{\alpha}} \to \infty$

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(3) If $\rho = \infty$ is included

logs are okay but $d_3 = 0$

b/c the power term goes to ∞ too quickly

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(4) Between 2 cylinders

full $C_1 = 0$ b/c of (1)

When you need all constants: