Green Functions

The Green function is the general solution [response, $x(t)$] to a given differential equation when the driving force is impulsive [delta function-like]. Since it is a <u>general</u> solution, the initial conditions [boundary conditions] are built into the Green function.

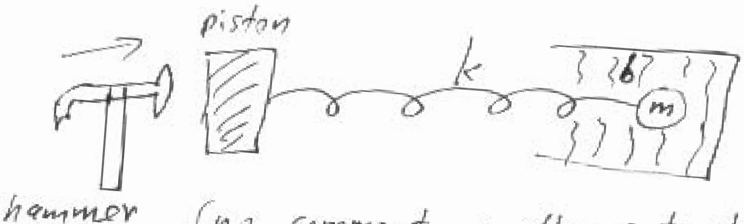
<u>e.g.</u> underdamped simple harmonic oscillator

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = \frac{F(t)}{m}$$

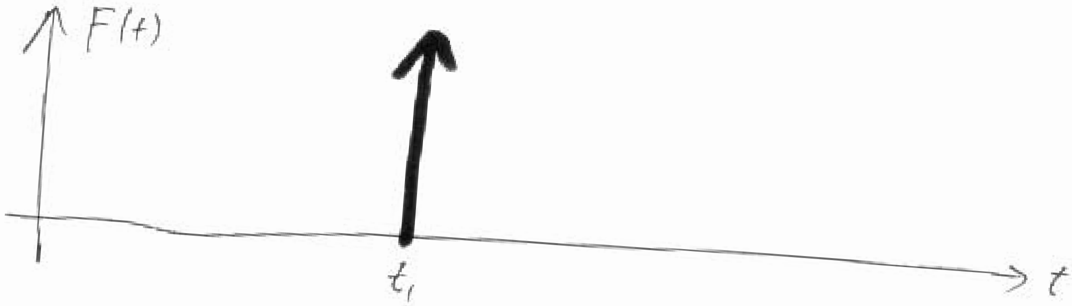
$\omega_0 \equiv \sqrt{\frac{k}{m}}$

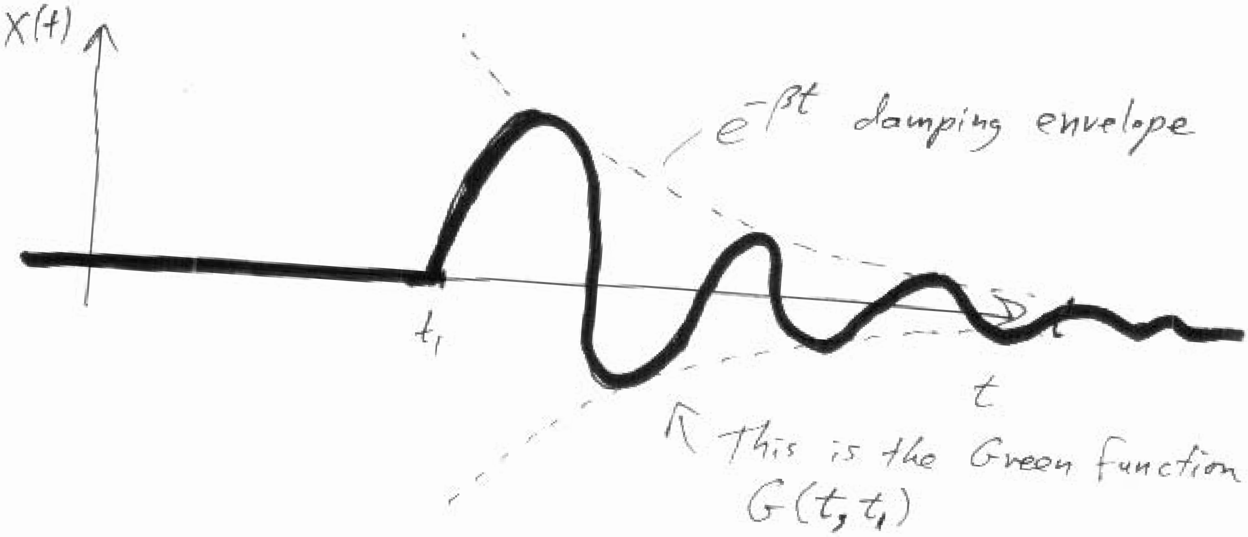
$2\beta \equiv \frac{b}{m}$

and $\beta < \omega_0$

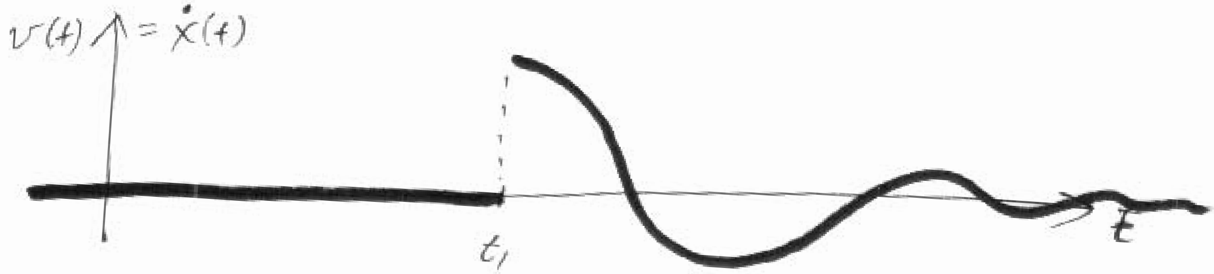


(no comments on the artwork, please!)





$G(t,t_1)$ is continuous, but the slope (velocity) changes at time t_1 discontinuously



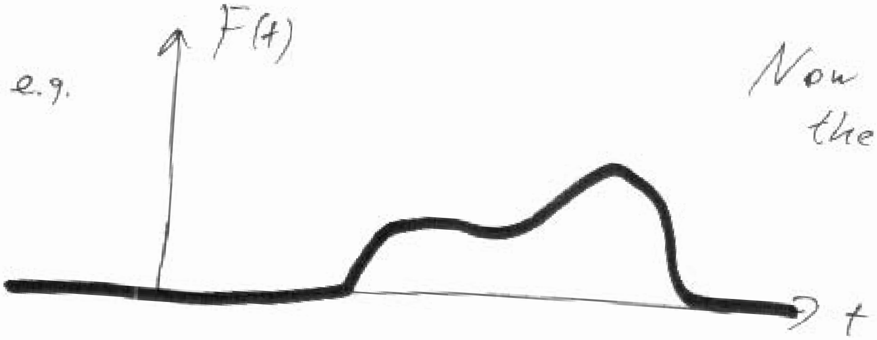
One more derivative will give a delta function in the acceleration $\ddot{x}(t)$. This is OK because Newton's 2^{nd} Law $F = ma$ implies an infinite acceleration when the Force is infinite.

Why is the Green function necessary?

Last time, the forcing function was $F(t) = F_0 \cos(\omega t)$ and we were able to guess the particular solution.

But what if $F(t)$ is some arbitrary function of time?

e.g.

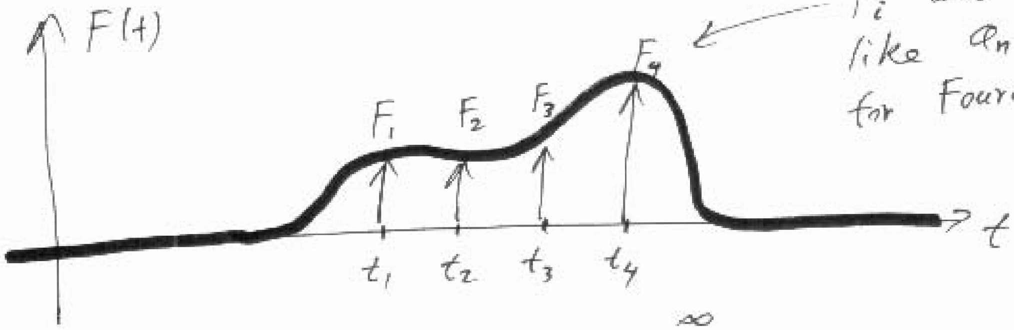


Now how do we find the general solution?

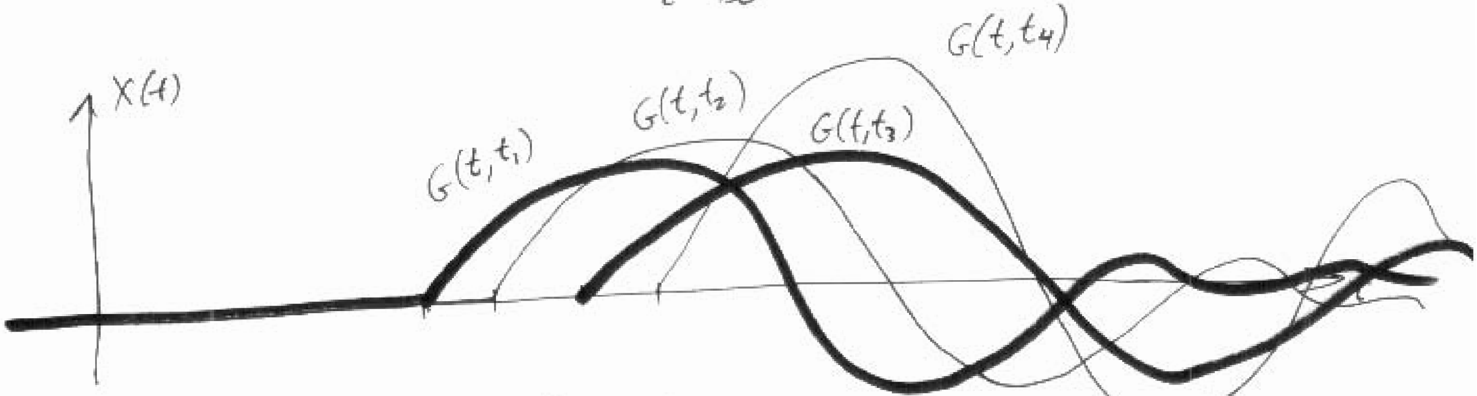
An arbitrary function $F(t)$ can be decomposed as:

1) sines + cosines (Fourier)
2) delta functions (Green)

F_i are amplitudes, like a_n and b_n for Fourier series



$$F(t) = \sum_{i=1}^{4} F_i\,\delta(t-t_i) \longrightarrow \int_{t'=-\infty}^{\infty} F(t')\,\delta(t-t')\,dt'$$



Superpose to get $X(t)$

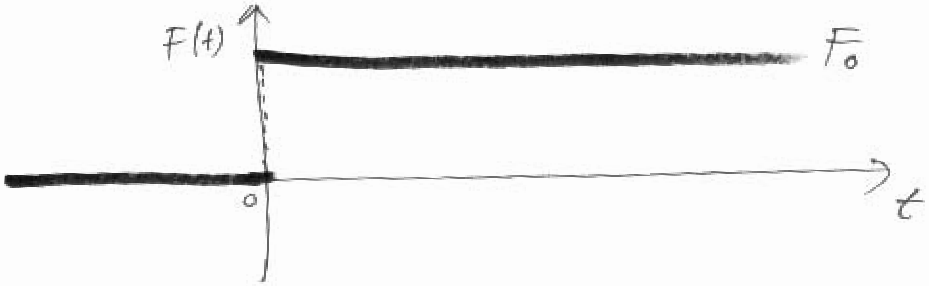
$$X(t) = \sum_{i=1}^{4} F_i\,G(t,t_i) \longrightarrow \int_{t'=-\infty}^{\infty} F(t')\,G(t,t')\,dt'$$

this is a convolution integral.

Great! Sort of... If only you knew $G(t,t')$ then this would give you $X(t)$ for any $F(t)$.

Derivation of $G(t,t')$

Consider this forcing function:



$$F(t) = F_0\,\theta(t)$$

We know the complementary solution for $t > 0$.

$$x_c(t) = e^{-\beta t}\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right]$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$

Guess a constant particular solution

$$x_p(t) = C \qquad \dot{x}_p(t) = 0 \qquad \ddot{x}_p(t) = 0$$

$$\ddot{x}_p(t) + 2\beta\,\dot{x}_p(t) + \omega_0^2\,x_p(t) = \frac{F(t)}{m} = \frac{F_0\,\theta(t)}{m} = \begin{cases} 0, & t<0 \\ \frac{F_0}{m}, & t>0 \end{cases}$$

$$0 + 0 + \omega_0^2\,C = \begin{cases} 0, & t<0 \\ \frac{F_0}{m}, & t>0 \end{cases}$$

$$x_p(t) = \begin{cases} 0, & t<0 \\ \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}, & t>0 \end{cases}$$

The general solution is $x(t) = x_c(t) + x_p(t)$

$$x(t) = \begin{cases} 0 & , \ t \leq 0 \\ e^{-\beta t}\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right] + \frac{F_0}{m\omega_0^2} & , \ t \geq 0 \end{cases}$$

determine A and B from initial conditions:

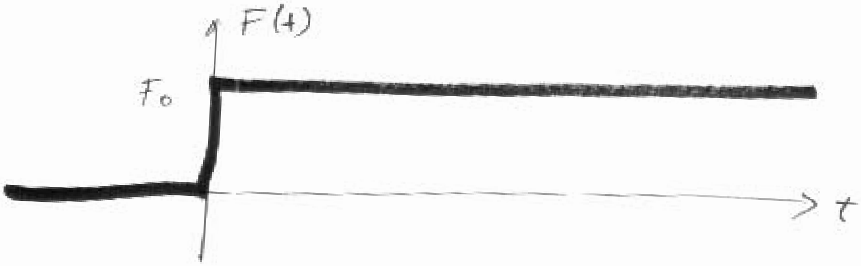
$$\left.\begin{array}{l} x(0) = 0 \\ v(0) = 0 \end{array}\right\} \text{ for example.}$$

$$0 = x(0) = A + \frac{F_0}{m\omega_0^2} \implies A = -\frac{F_0}{m\omega_0^2}$$

$$v(t) = \dot{x}(t) = \begin{cases} 0 & , \ t \leq 0 \\ e^{-\beta t}\left\{-\beta\left[A\cos(\omega_1 t) + B\sin(\omega_1 t)\right] + \omega_1\left[-A\sin(\omega_1 t) + B\cos(\omega_1 t)\right]\right\} & , \ t \geq 0 \end{cases}$$

$$0 = v(0) = -\beta A + \omega_1 B \implies B = \frac{\beta A}{\omega_1} = -\frac{\beta F_0}{m\omega_1\omega_0^2}$$

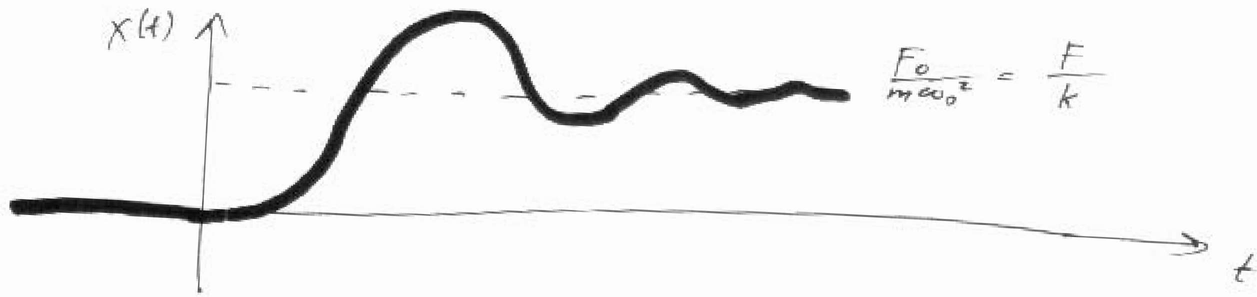
So for this force



$$F(t) = F_0\,\theta(t)$$

the general solution is

$$X(t) = \begin{cases} 0, & t \le 0 \\ \dfrac{F_0}{m\omega_0^2}\left[1 - e^{-\beta t}\cos(\omega_1 t) - \dfrac{\beta}{\omega_1} e^{-\beta t}\sin(\omega_1 t)\right], & t \ge 0 \end{cases}$$

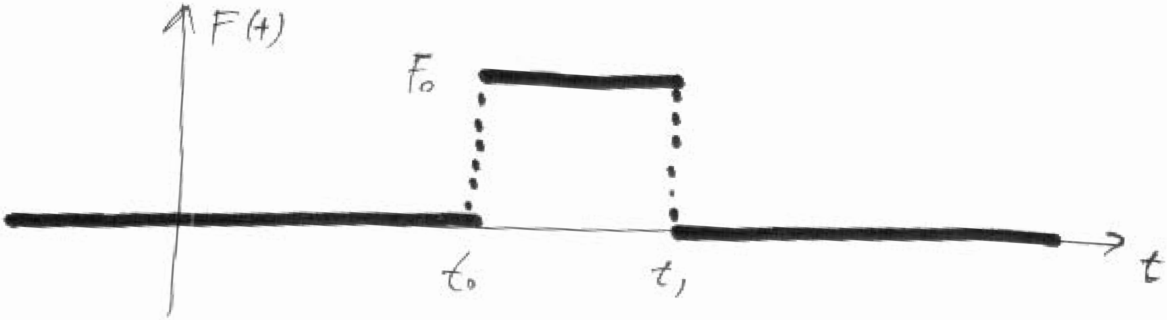


The dashed line is the particular solution $X_p(t)$ which is the steady state (long time) solution.
The complementary solution is transient and the damping envelope $e^{-\beta t}$ will effectively wipe them out in 5 e-folding times.

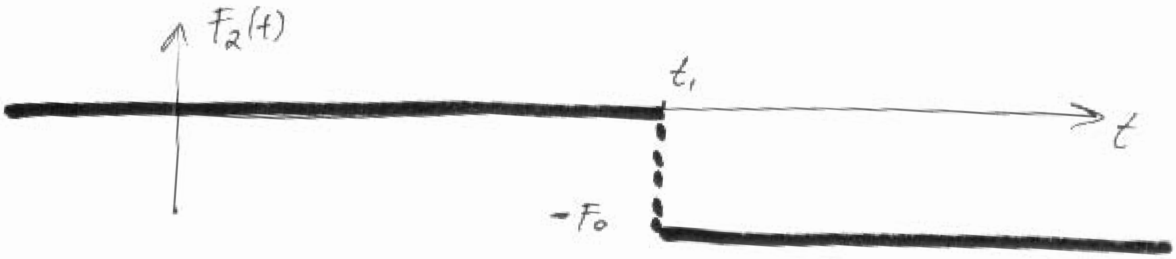
Now, what if the constant force F_0 turns on at t_0 instead of $t = 0$?

Then in the general solution above, just replace t (really $t-0$) with $(t-t_0)$. So the answer is $X(t-t_0)$.

How about this force?



This is the superposition of two forces $F_1(t)$ and $F_2(t)$.



But we know the general solution for each of these

$$X_1(t) = X(t-t_0) \qquad\qquad X_2 = -X(t-t_1)$$

So we superpose the two solutions (isn't linearity great?)

The general solution with square pulse forcing is

$$X(t-t_0) - X(t-t_1)$$

Now let $(t_1 - t_0) \to 0$ while $F_0 \to \infty$ such that the area under the square pulse is constant, $F_0(t_1 - t_0) = \text{const.}$ The force will become a delta function. Under the same limits, the general solution will become the Green function.