homogeneous – there is no term without a derivative of y (including the 0^{th}).

non-homogeneous example:

$$y'(x) - y(x) = 7$$

a first-order, linear, ordinary differential eq. (as opposed to a partial diff eq.)

& solve:

goal: find $y(x) = \begin{pmatrix}\text{solution to} \\ \text{homogeneous} \\ \text{eq}\end{pmatrix} + \begin{pmatrix}\text{particular} \\ \text{solution}\end{pmatrix}$

$$y = y_h + y_p$$

$$y_h = y'(x) - y(x)$$

$$y(x) = \underbrace{Ae^x}_{y_h} + \underbrace{-7}_{y_p,\ \text{guess}}$$

take derivative

$$y'(x) = Ae^x + 0$$

$$y'(x) - y(x) = Ae^x - (Ae^x - 7) = 7 \checkmark$$

$\underline{y(x) = Ae^x - 7}$ is the general soln

2nd Order Differential Equations:

$$y''(x) + y(x) = 0$$

2nd order, linear, homogeneous & ordinary diff eq

⇒ expect 2 arbitrary constants b/c it's a 2^{nd} order eq.

Guess: $y(x) = Ae^x \quad y''(x) = Ae^x \rightarrow Ae^x + Ae^x = 2Ae^x \neq 0$

Guess: $y(x) = A\sin x \quad y'(x) = -A\cos x \quad y''(x) = -A\sin x$

$\therefore -A\sin x + A\sin x = 0 \checkmark$

$\therefore y(x) = A\sin x \rightarrow$ one arbitrary constant

Guess: $y(x) = B\cos x \rightarrow$ also works...

General Soln: $A\sin x + B\cos x$

$y(x) = C\sin(x + D)$ ↩ equivalent

other forms of the solution:

$$y(x) = E\cos(x+F)$$

& $y(x) = Ge^{ix} + He^{-ix}$

$$e^{ix} = \cos(x) + i\sin(x)$$
$$e^{-ix} = \cos(x) - i\sin(x)$$

$\cos(x) = \dfrac{e^{ix}+e^{-ix}}{2}$ & $\sin(x) = \dfrac{e^{ix}-e^{-ix}}{2i}$

$y''(x) + y(x) = x^2$ 2nd order, linear, nonhomogeneous & ordinary

Need to determine the particular solution:

<u>Guess</u>: $y(x) = ax^2 + bx + c$ a general quadratic

$y'(x) = 2ax + b$ $y''(x) = 2a$

$y''(x) + y(x) = 2a + ax^2 + bx + c = x^2$

x^2 terms: $a = 1$ ✓

x " $b = 0$ ✓

x^0 " $2a + c = 0$ $c = -2$ ✓

General Solution $y(x) = y_h + y_p$

$$\boxed{y(x) = \underbrace{A\sin(x) + B\cos(x)}_{y_h} + \underbrace{x^2 - 2}_{y_p}}$$

"Studying non-linear systems is like studying non-elephant biology"

— Stanislaw Ulam

- chaos implies non-linearity but non linearity doesn't imply chaos -

$f''(t) = t$ second-order, linear, ordinary, non-homogeneous...

b/c it's not $f(t)$...

$f_g(t) = f_h(t) + f_p(t)$

(general) (homogeneous) (particular)

$\underline{f_h(t)}$:

$f''(t) = 0 \qquad f_h(t) = At + B$

$\underline{f_p(t)}$: guess: $f_p(t) = at^2 + bt + c$

$f'_p(t) = 2at + b$

$f''_p(t) = 2a$

$\Rightarrow$ all coefficients are zero, so none of those belong...

A better guess: $f_p(t) = at^3$

$f'_p(t) = 3at^2$

$f''_p(t) = 6at \qquad 6at = t \qquad 6a = 1 \quad a = \frac{1}{6}$

$\therefore f_p(t) = \frac{1}{6}t^3$

$\underline{f_g(t) = f_h(t) + f_p(t) = At + B + \frac{1}{6}t^3}$

Initial Conditions:

"Boundary Conditions"

EX:

$\left.\begin{array}{l} f(0) = 2 \\ f'(0) = 7 \end{array}\right\}$ use to solve for A & B

$f_g(t) = At + B + \frac{1}{6}t^3 \Rightarrow f(0) = 0 + B + 0 = \underline{B = 2}$

$f'(t) = A + \frac{1}{2}t^2 \Rightarrow f'(0) = A + 0 = \underline{A = 7}$

$\underline{f_g(t) = 7t + 2 + \frac{1}{6}t^3}$

— Damped Simple Harmonic Oscillators —

→ imagine linear resistive force: $\vec{F}_{\text{viscous drag}} = -b\vec{v}$

$F(x) = -kx - b\dot{x}$ new velocity term

$v = \dot{x} = \frac{dx}{dt}$

$a = \ddot{x} = \frac{d^2x}{dt^2}$

$m\ddot{x} = -kx - b\dot{x}$

$\ddot{x} = -\frac{k}{m}x - \frac{b}{m}\dot{x}$ ($\frac{b}{m} = 2\beta$)

ω_0^2 = undamped oscillation frequency

$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$

2nd order, linear (in x), homogeneous (in x) & ordinary

adding on any function of t or constant (doesn't include an x) would make it non-homogeneous!

To solve this: 2nd order, so 2 constants

guess: $x(t) = Ae^{rt}$ *doesn't hurt to "overguess" (throw in more arbitrary constants than necessary)

$\dot{x}(t) = rAe^{rt}$

$\ddot{x}(t) = r^2Ae^{rt}$

$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$

$r^2Ae^{rt} + 2\beta rAe^{rt} + \omega_0^2 Ae^{rt} = 0$

b/c $A \neq 0$, you can divide through by A, as well as the exponential, e^{rt}, leaving you:

$$r^2 + 2\beta r + \omega_0^2 = 0$$

characteristic eq'n

quadratic formula $r_\pm = -\frac{2\beta \pm \sqrt{\beta^2 - \omega_0^2}}{2(1)} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$ (4?)

general solution to damped SHO

$$x(t) = A_+e^{r_+t} + A_-e^{r_-t} = A_+e^{(-\beta+\sqrt{\beta^2-\omega_0^2})t} + A_-e^{(-\beta-\sqrt{\beta^2-\omega_0^2})t}$$

just to differentiate; could be A_1 & A_2, etc

factor out $e^{-\beta t}$ $\quad x(t) = e^{-\beta t}\left[A_+e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_-e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$

should be just t?

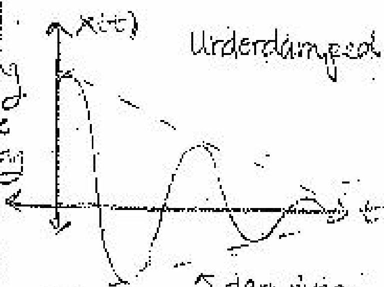
3 cases depending on magnitudes of β^2 & ω_0^2

Underdamped
Case I: $\beta < \omega_0$ so $\sqrt{\beta^2-\omega_0^2}$ is imaginary
$\Rightarrow$ expect oscillations (solutions looks like sin's & cos's)
β is the damping coefficient

Case II: $\beta = \omega_0$ so $\sqrt{\beta^2-\omega_0^2} = 0$
Critically damped – no oscillation

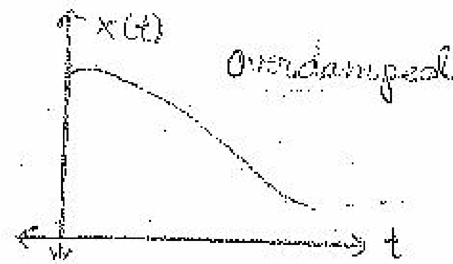
Case III: $\beta > \omega_0$ so $\sqrt{\beta^2-\omega_0^2}$ is real
Overdamped (no oscillation)

*all starting w/ positive velocity



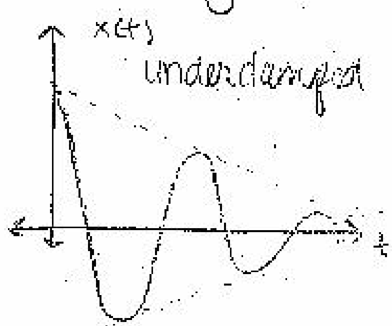
← damping envelope $\sim e^{-\beta t}$

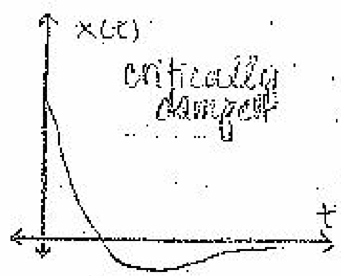
goes to zero fastest

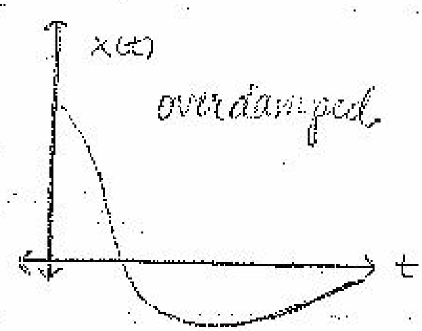


ex: any musical instrument or music in general
ex: shock absorbers
ex. door closing

*now w/ negative velocity







Case One: underdamped $\beta < \omega_0$

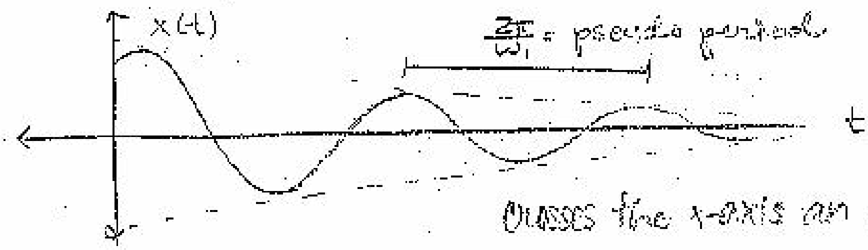
define $\omega_1^2 = \omega_0^2 - \beta^2 > 0$ ← take root of neg # ←

ω_1 is real

Soln: $x(t) = A_+ e^{r_+ t} + A_- e^{r_- t} = e^{-\beta t}\left[A_+ e^{i\omega_1 t} + A_- e^{-i\omega_1 t}\right]$

$= e^{-\beta t}\left[B\sin(\omega_1 t) + C\cos(\omega_1 t)\right]$

$= e^{-\beta t}\left[D\sin(\omega_1 t + E)\right]$



never "really" repeats

crosses the t-axis an infinite # of times

Case Two: critically damped $\beta = \omega_0$

$r_+ = -\beta = r_-$ → 2 identical roots (like undamped case)

degenerate roots of characteristic equation

problem: $A_+ e^{-\beta t}$ & $A_- e^{-\beta t}$ are no longer linearly independent

So, take one solution & multiply by powers of t, until you get linearly independent set

why? $e^{\sqrt{\beta^2-\omega_0^2}\,t}$ → Taylor expand around

small, ignore...

$e^{\sqrt{\beta^2-\omega_0^2}\,t} = 1 + t\sqrt{\beta^2-\omega_0^2} + t^2(\ldots) + \ldots$

$x(t) = \underbrace{A_1 e^{-\beta t}}_{\text{guaranteed to solve diffeq}} + A_2 e^{-\beta t}\,t$ then plug this in to diffeq to see if it works

if this doesn't work, try higher powers of t → $x_2(t) = A_2 t e^{-\beta t}$

$\dot{x}_2(t) = A_2 e^{-\beta t}(1 - \beta t)$

$\ddot{x}_2(t) = A_2 e^{-\beta t}(-2\beta + \beta^2 t)$ & plug these into diffeq.

$\omega_0^2 = \beta^2$ replaced

t

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$$A_2 e^{-\beta t}(-2\beta + \beta^2 t) + 2\beta A_2 e^{-\beta t}(1-\beta t) + \omega_0^2 A_2 e^{-\beta t} = 0$$

$\Rightarrow$ divide through by $A_2 e^{-\beta t}$ b/c $\neq 0$

$$= -2\beta + \beta^2 t + 2\beta - 2\beta^2 t + \beta^2 t = 0 \checkmark$$

Case 3: Overdamped $\beta > \omega_0$

define $\omega_2^2 = \beta^2 - \omega_0^2 > 0$ $\therefore$ ω_2 is real

$$x(t) = \underbrace{e^{-\beta t}}_{\text{damping envelope}}\left[A_+ e^{\overbrace{+\omega_2 t}^{\text{exponential growth}}} + A_- e^{\overbrace{-\omega_2 t}^{\text{exponential decay}}}\right]$$

Damped Driven Simple Harmonic Motion

$\Sigma F = ma$

$$-kx(t) - b\dot{x}(t) + F_0\cos(\overbrace{\omega}^{\text{driving frequency}} t) = m\ddot{x}(t)$$

driving force w/ frequency ω

creates drag, damping

$\omega_0 = \sqrt{\frac{k}{m}}$

ω_1 = pseudo frequency $= \sqrt{\omega_0^2 - \beta^2}$ (for underdamped)

$\omega_2 = \sqrt{\beta^2 - \omega_0^2}$ overdamped motion

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = A\cos(\omega t)\ , \text{ where } A = \frac{F_0}{m}$$

a linear, 2nd order, homogeneous, ordinary equation

general solution = complementary solution + particular solution

complementary solution = homogeneous solution (RHS = 0)

transients that die off

w/ 2 arbitrary constants fixed by initial conditions... boundary conditions could've also been written as $E\cos(\omega t) + F\sin(\omega t)$

particular solution → no arbitrary constants ~ a guess based on the driving force ~ steady state solution (all that's left after you wait long enough)

$$x_c(t) = e^{-\beta t}\left[A_+ e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_- e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

$$x_p(t) = \text{guess} = D\cos(\omega t - \delta) \rightarrow \text{phase shift}$$

* amplitude of guess is always D, just historically * → humor them

→ solve for D & δ → no arbitrariness in these!

ASIDE $\ddot{x}_c(t) + 2\beta\dot{x}_c(t) + \omega_0^2 x_c(t) = 0$ b/c it's the homogeneous solution

$$\dot{x}_p(t) = -D\omega\sin(\omega t - \delta)$$

$$\ddot{x}_p(t) = -D\omega^2\cos(\omega t - \delta)$$

& plug into $\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = A\cos(\omega t)$

$$\ddot{x}_p(t) + 2\beta\dot{x}_p(t) + \omega_0^2 x_p(t) = A\cos(\omega t)$$

$$D\left[-\omega^2\cos(\omega t - \delta) - 2\beta\omega\sin(\omega t - \delta) + \omega_0^2\cos(\omega t - \delta)\right] = A\cos(\omega t)$$

rewrite using angle difference formula

$\cos(\omega t)\cos(\delta) + \sin(\omega t)\sin\delta$ $\quad$ $\sin(\omega t)\cos(\delta) - \cos(\omega t)\sin(\delta)$

& a lot of algebra... until...

ANSWER: (well, almost)

$$\left\{A - D\left[(\omega_0^2 - \omega^2)\cos(\delta) + 2\omega\beta\sin(\delta)\right]\right\}\cos(\omega t)$$
$$- D\left[(\omega_0^2 - \omega^2)\sin(\delta) - 2\omega\beta\cos(\delta)\right]\sin(\omega t) = 0 \quad \forall t$$

for all t

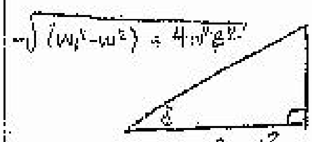
cos(wt) & sin(wt) parts must equal 0 independently (don't let D = 0!)

→ 2nd line: $\left[(\omega_0^2 - \omega^2)\sin(\delta) - 2\omega\beta\cos(\delta)\right] = 0$

$$\frac{\sin(\delta)}{\cos(\delta)} = \frac{2\omega\beta}{\omega_0^2 - \omega^2} = \tan(\delta) = \frac{opp}{adj}$$

ARCTAN

$\delta = \arctan\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)$, then plug back into first line...



taking the sin/cos of arctan's of something

$\therefore \sin(\delta) = \frac{opp}{hyp} = \frac{2\omega\beta}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta^2}}$

$\&\ \cos(\delta) = \frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta^2}}$

1st line:

$$D = \frac{A}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\omega^2\beta^2}}$$

General Solution:

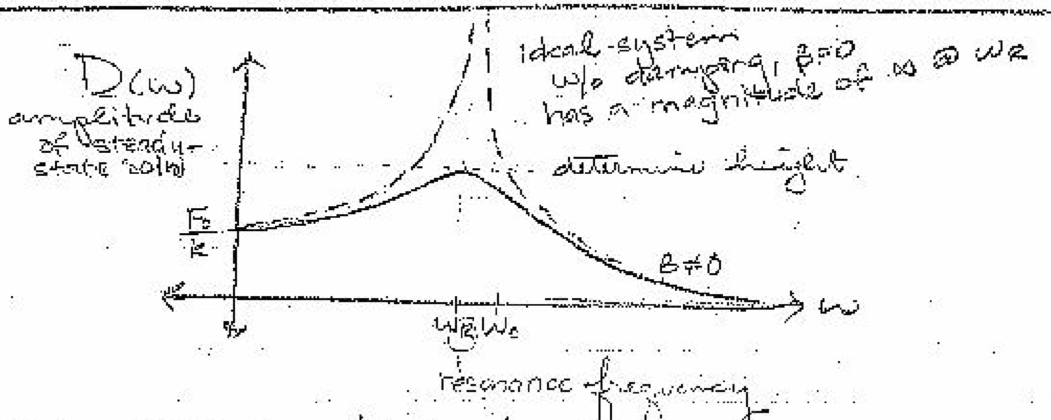
$x_c(t)$ = transient

$$x(t) = e^{-\beta t}\left[A_+ e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_- e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

(A) $x_p(t)$ = steadystate

$$+ \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}} \cos\left[\omega t - \arctan\left(\frac{2\omega\beta}{\omega_0^2-\omega^2}\right)\right]$$

* after 5 pseudo periods (ω_1), only the 2nd line will remain



when $\omega = 0$ (no driving force)

$$D = \frac{F_0/m}{\omega_0^2} = \frac{F_0}{k}$$

$$\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$$

How to prove This? let $\left.\frac{dD(\omega)}{d\omega}\right|_{\omega=\omega_R} = 0$ — make it equal zero!

Phase Shift