$$\dot{\hat{e}}_r = \frac{\cancel{x^2\dot{x}\hat{e}_x} + x^2\dot{y}\hat{e}_y + y^2\dot{x}\hat{e}_x + \cancel{y^2\dot{y}\hat{e}_y} - [\cancel{x^2\dot{x}\hat{e}_x} + xy\dot{x}\hat{e}_y + xy\dot{y}\hat{e}_x + \cancel{y^2\dot{y}\hat{e}_y}}{(x^2+y^2)^{3/2}}$$

$$\dot{\hat{e}}_r = \frac{x^2\dot{y}\hat{e}_y + y^2\dot{x}\hat{e}_x - xy\dot{x}\hat{e}_y - xy\dot{y}\hat{e}_x}{(x^2+y^2)^{3/2}} \rightarrow \text{factors } (1)$$

$$\dot{\hat{e}}_r = \frac{(-y\hat{e}_x + x\hat{e}_y)(x\dot{y} - y\dot{x})}{(x^2+y^2)^{3/2}}$$

$$\dot{\hat{e}}_r = \frac{(-y\hat{e}_x + x\hat{e}_y)}{\sqrt{x^2+y^2}} \cdot \frac{(x\dot{y} - y\dot{x})}{(x^2+y^2)}$$

$$\dot{\hat{e}}_r = \hat{e}_\theta \dot{\theta}$$

For homework,
Show that $\dot{\hat{e}}_\theta = -\hat{e}_r \dot{\theta}$
using the same procedure as seen above

Cartesian $\vec{r} = x\hat{e}_x + y\hat{e}_y$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y$$

2D Polar: $\vec{r} = r\hat{e}_r$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[r\hat{e}_r] = \left(\frac{dr}{dt}\right)\hat{e}_r + r\frac{d\hat{e}_r}{dt}$$

$$\vec{v} = \underbrace{\dot{r}}_{\text{radial velocity}}\hat{e}_r + \underbrace{r\dot{\theta}}_{\text{tangential velocity}}\hat{e}_\theta \text{ from above} = V_r\hat{e}_r + V_\theta\hat{e}_\theta$$

future homework is to find acceleration by going one more step

Oct 24th → In class Midterm
3 questions: easy, medium, hard

Generalized Coordinates (3D)
(q_1, q_2, q_3) e.g. Cartesian Coordinates (x, y, z)
Spherical Polar (r, θ, φ)
Cylinder Polar (ρ, φ, z)
same φ

* Cylindrical is kinda a cross btwn cartesian & Spherical polar coordinates

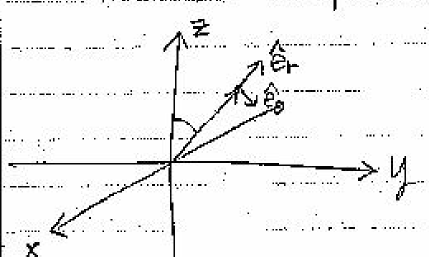
unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$
orthonormal $\hat{e}_n \cdot \hat{e}_m = \delta_{nm}$

Scalar functions:
temprature: $T(q_1, q_2, q_3)$

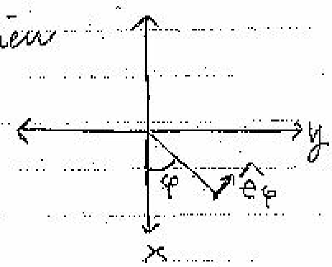
Vector functions: $\vec{V} = \begin{pmatrix} V_1(q_1, q_2, q_3) \\ V_2(q_1, q_2, q_3) \\ V_3(q_1, q_2, q_3) \end{pmatrix}$

where V_1 is the component of $\vec{V}$ in the direction of increasing q_1, that is $\hat{e}_1$

$$V_1 = \vec{V} \cdot \hat{e}_1$$



& a top view



The unit vectors are themselves functions of coordinates $\hat{e}_1 = \hat{e}_1(q_1, q_2, q_3)$

Scale Functions (h_1, h_2, h_3)

$h_1(q_1, q_2, q_3)$

In Cartesian, $h_x = 1$, $h_y = 1$, $h_z = 1$, that's why we never noticed them

* On homework *

Line Element $\vec{ds}$

$$\vec{ds} \cdot \vec{ds} = |\vec{ds}|^2 = ds^2$$

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$$

Cartesian: $\vec{ds} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z$

$$ds^2 = dx^2 + dy^2 + dz^2$$

$\therefore$ in Cartesian, you can see $\underline{h_i = 1}$

Volume Element

$\iiint f(q_1, q_2, q_3)\, d^{3}V$ — only notation for some books...

In Cartesian: $dV = dx\,dy\,dz$

In Generalized Coordinates: $dV = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$

Ex. Spherical $dV = r^2 \sin\theta\, dr\, d\theta\, d\varphi$

derived on next page

(we'll hafta do it for cylindrical coordinates)

Spherical: $dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$

$h_1 = 1 \quad h_\theta = r \quad h_\varphi = r\sin\theta$

definitions: where θ = polar angle $[0, \pi]$

$x = r\sin\theta\cos\varphi$ & φ = azimuthal angle $[0, 2\pi]$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$ (component of r along z-axis)

* different mathematical definition

(using product rule)

$dx = dr\sin\theta\cos\varphi + r(\cos\theta \, d\theta)\cos\varphi + r\sin\theta(-\sin\varphi \, d\varphi)$

$dy = dr\sin\theta\sin\varphi + r(\cos\theta \, d\theta)\sin\varphi + r\sin\theta\cos\varphi \, d\varphi$

$dz = dr\cos\theta - r\sin\theta \, d\theta$

* square each of these & add them together & you get $d\underline{s}^2$!

& $d\underline{s}^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\varphi^2$

$h_1 = 1 \quad h_2 = r \quad h_\varphi = r\sin\theta$

When doing cylindrical terms:

$dV = h_\rho h_\varphi h_z \, d\rho \, d\varphi \, dz$ same as in cylindrical

Physical Example →

$\Phi(\vec{r})$ is the electric scalar potential aka, voltage ☺ of a pt charge @ the origin

Cartesian $\Phi(\vec{r}) = \Phi(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2+z^2}}$

Spherical $\Phi(\vec{r}) = \Phi(r, \theta, \varphi) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

* Spherical is easier

Electric Field (vector function of coordinates

$\vec{E}(\vec{r}) = -\nabla\Phi(\vec{r})$

gradient

pts in direction of fastest increasing height what

Cartesian: $\vec{E}(\vec{r}) = \begin{pmatrix} E_x(x,y,z) \\ E_y(x,y,z) \\ E_z(x,y,z) \end{pmatrix}$

$$E_x = -\frac{\partial}{\partial x}\Phi(x,y,z) = -\frac{\partial}{\partial x}\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{x^2+y^2+z^2}}$$

Spherical Coordinates (Polar):

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) = \begin{pmatrix} E_r(r,\theta,\varphi) \\ E_\theta(r,\theta,\varphi) \\ E_\varphi(r,\theta,\varphi) \end{pmatrix}$$

$E_\theta = 0 \quad E_\varphi = 0$

$$E_r = -\frac{\partial}{\partial r}\Phi(r,\theta,\varphi) = -\frac{\partial}{\partial r}\frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

$$E_r = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

$$\vec{E}(\vec{r}) = \begin{pmatrix} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \\ 0 \\ 0 \end{pmatrix}$$

Generalized Functions (Distributions) bis

(latin for again)

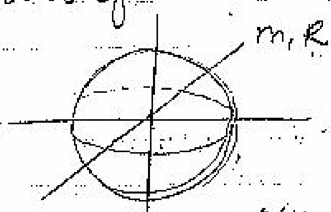
Reminder: $\delta(\alpha)$ has dimension $\frac{1}{\alpha}$

$$\int_{-\infty}^{\infty}\delta(\alpha)\,d\alpha = 1$$

eg. $\delta(x)$ dimension $\left(\frac{1}{\text{length}}\right)$
$\delta(\theta)$ dimension (1)
must cancel w/ units of x & θ, so that after this integration you get a dimensionless 1

$D(\vec{r})$ = volume mass density : dimensions $\frac{\text{mass}}{\text{length}^3}$
scalar function of position

eg. solid sphere



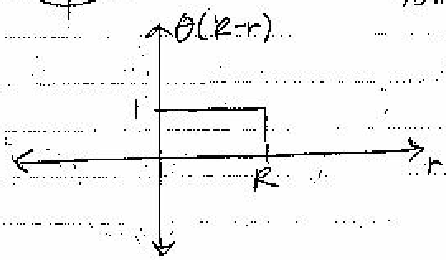
spherical coordinate ↓ ; radius of sphere ↓

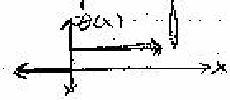
$$D(\vec{r}) = \begin{cases} \frac{m}{4/3\,\pi R^3} & , r < R \\ 0 & , r > R \end{cases}$$

outside sphere

$$D(\vec{r}) = \frac{m}{4/3\,\pi R^3}\,\theta(R-r)$$ (no dimensions)

heaviside function





Spherical Polar: $D(\vec{r}) = \frac{m}{4/3\,\pi R^3}\,\theta(R-r)$

Cartesian: $D(\vec{r}) = \frac{m}{4/3\,\pi R^3}\,\theta\left(R-\sqrt{x^2+y^2+z^2}\right)$

Cylindrical Polar: $D(\vec{r}) = \frac{m}{4/3\,\pi R^3}\,\theta\left(R-\sqrt{\rho^2+z^2}\right)$

eg. Point Mass at Origin in Cartesian

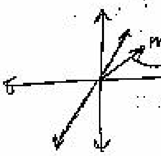
$D(\vec{r}) = m\,\delta(x)\delta(y)\delta(z) \Rightarrow \iiint D(\vec{r})\,dV = m$ (over all space)

(density is zero everywhere except @ the origin where it is infinite)

$$\iiint_{\text{all space}} D(\vec{r})\,dV = m = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty}\int_{z=-\infty}^{\infty} m\,\delta(x)\delta(y)\delta(z)\,dx\,dy\,dz$$

$$= m\underbrace{\int_{x=-\infty}^{\infty}\delta(x)dx}_{1}\;\underbrace{\int_{y=-\infty}^{\infty}\delta(y)dy}_{1}\;\underbrace{\int_{z=-\infty}^{\infty}\delta(z)dz}_{1} = m$$

Related Example:



$\vec{r}\,' = (x', y', z')$ $\quad D(\vec{r}) = m\,\delta(x-x')\,\delta(y-y')\,\delta(z-z')$

& now in Spherical Polar: for a pt @ the origin

$D(\vec{r}) = m\,\delta(\vec{r})\,f(r)$

↳ some function of r, w/ no dependence on θ, φ dependence

$$\iiint_{\text{All Space}} D(\vec{r})\,dV = m = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} m\,\delta(r)f(r)\sin\theta\, r^2\,dr\,d\theta\,d\varphi$$

$$= m\int_{\varphi=0}^{2\pi} d\varphi\int_{\theta=0}^{\pi}\sin\theta\,d\theta\int_{r=0}^{\infty}\delta(r)f(r)r^2\,dr$$

→ must get rid of this r^2 so $f(r)$ must have $\frac{1}{r^2}$

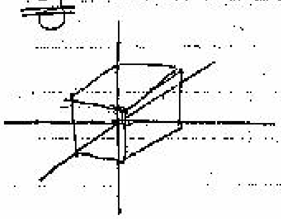
*if you integrate over φ & θ, you get 4π

$D(\vec{r}) = m(2\pi)(2)$

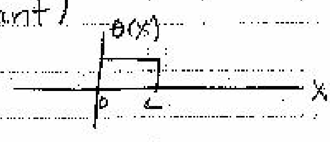
$f(r) = \frac{1}{4\pi r^2}$ get rid of r^2, get rid of 4π

$D(\vec{r}) = \frac{m\,\delta(\vec{r})}{4\pi r^2}$ $\quad$ Cylindrical for Homework

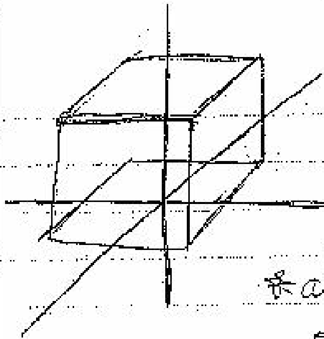
eg. Solid Cube: one corner @ origin (1st octant)



$$D(\vec{r}) = \begin{cases} \frac{m}{L^3}, & 0<x<L,\ 0<y<L,\ 0<z<L \\ 0, & \text{otherwise} \end{cases}$$



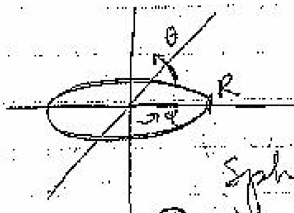
$$D(\vec{r}) = \frac{m}{L^3}\,\underbrace{\theta(x)\theta(L-x)}_{\text{also: }\theta(x)-\theta(x-L)}\,\theta(y)\theta(L-y)\,\theta(z)\theta(L-z)$$



Cube centered around origin

$$D(\vec{r}) = \frac{m}{L^3}\,\theta\left(\tfrac{L}{2}-|x|\right)\theta\left(\tfrac{L}{2}-|y|\right)\theta\left(\tfrac{L}{2}-|z|\right)$$

* as long as they are between $\pm\frac{L}{2}$ from origin, it's on... thus the cube



eg. Ring mass m, radius R in the x-y plane

Spherical polar — b/c it doesn't depend on φ

$$D(\vec{r}) = m\,\delta(\theta-\tfrac{\pi}{2})\,\delta(r-R)\,f(r,\theta)$$

theta, not heavyside ☺

& no φ dependence!!

$$\iiint_{\text{all space}} D(\vec{r})\,dV = m = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{\infty} m\,\delta(\theta-\pi/2)\,\delta(r-R)\,f(r,\theta)\,r^2\sin\theta\,d\ldots$$

no θ dependence anymore...

$$m = m\int_{\varphi=0}^{2\pi} d\varphi \int_{\theta=0}^{\pi}\underbrace{\delta(\theta-\tfrac{\pi}{2})\sin\theta\,d\theta}_{\sin(\frac{\pi}{2})} \int_{r=0}^{\infty}\delta(r-R)\,f(r)$$

$$f(r) = \frac{1}{2\pi r^2}$$

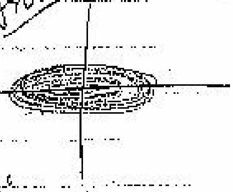
$$m = m\,(2\pi)(1)\left(\frac{1}{2\pi}\right)$$

$$f(r) = \frac{1}{2\pi r^2}$$

$$D(\vec{r}) = \frac{m\,\delta(\theta-\pi/2)\,\delta(r-R)}{2\pi r^2}$$

the delta fixes this to "R" (only under the integral sign)

Grad Problem



disk mass m, Radius R in $z=0$ plan

$$\int f(r)\,dr = 2\pi$$

Trick: integrate over r from 0 to $\frac{R}{2}$

expect: $\frac{m}{4}$ → this'll help you fix f

Differential Operators

① Gradient $\vec{\nabla}$ (nabla)

acts on a scalar function of position, $f(\vec{r})$

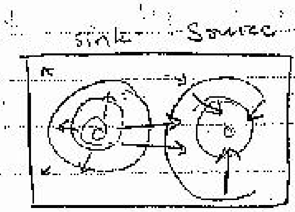
& produces a vector function of position

$$\vec{\nabla} f(\vec{r}) = \text{grad}\,[f(\vec{r})]$$

$\vec{\nabla} f(\vec{r})$ points in the direction of fastest increase of $f(\vec{r})$, magnitude of $\vec{\nabla} f(\vec{r})$ tells you exactly how fast $f(\vec{r})$ is changing

examples: temperature in a room w/ a match (grad points towards increasing temperature) Also, a topographical map points uphill in the fasted increasing direction

*When you ~~are~~ are the highest point, grad function is 0



<u>Cartesian</u> $f(x,y,z) = f(\vec{r})$

$$\vec{\nabla} f(x,y,z) = \left[\frac{\partial}{\partial x} f(x,y,z)\right]\hat{e}_x + \left[\frac{\partial}{\partial y} f(x,y,z)\right]\hat{e}_y + \left[\frac{\partial}{\partial z} f(x,y,z)\right]\hat{e}_z$$

eg. $f(x,y,z) = x^2 \sin(y) + \cos(z)$ → scalar function, no vectors

*when taking partial derivatives, treat other variables as constants

$\frac{\partial}{\partial x} f(x,y,z) = 2x \sin(y)$ $\quad\quad$ $\frac{\partial}{\partial z} f(x,y,z) = -\sin(z)$

$\frac{\partial}{\partial y} f(x,y,z) = x^2 \cos y$

$$\vec{\nabla} f(x,y,z) = \begin{pmatrix} 2x\sin(y) \\ x^2\cos(y) \\ -\sin(z) \end{pmatrix} = \begin{pmatrix} \vec{\nabla} f_x \\ \vec{\nabla} f_y \\ \vec{\nabla} f_z \end{pmatrix}$$

In general, (q_1, q_2, q_3)

Scalar function $\Phi(q_1, q_2, q_3)$ → like voltage (a traditional scalar function apparently)

$$\vec{\nabla}\Phi(q_1,q_2,q_3) = \frac{1}{h_1}\frac{\partial\Phi}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial q_3}\hat{e}_3$$

& spherical polar

$h_r = 1 \quad h_\theta = r \quad h_\varphi = r\sin\theta$

Scalar function: $\Phi(r, \theta, \varphi)$

$$\vec{\nabla}\Phi(r,\theta,\varphi) = \frac{\partial\Phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\hat{e}_\varphi$$

Cylindrical Polar $h_\rho = 1,\ h_\varphi = \rho,\ h_z = 1$

$$\Phi(\rho, \varphi, z) \qquad \vec{\nabla}\Phi(\rho,\varphi,z) = \frac{\partial\Phi}{\partial\rho}\hat{e}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\varphi}\hat{e}_\varphi + \frac{\partial\Phi}{\partial z}\hat{e}_z$$

② Divergence $\vec{\nabla}\cdot$

acts on a vector function of position $\vec{V}(\vec{r})$ & produces a scalar function of position

$$\vec{\nabla}\cdot V(\vec{r}) = \mathrm{div}[\vec{V}(\vec{r})]$$

Think of $\vec{V}(\vec{r})$ as a velocity field of a fluid, then $\vec{\nabla}\cdot\vec{V}(\vec{r})$ describes the net flow out of a small region around $\vec{r}$