Generalized Coordinates $\{q_1, q_2, q_3\}$

unit vectors : $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

orthonormal $\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$\hat{e}_i$ point in the direction of increasing q_i.

Scale functions $\{h_1, h_2, h_3\}$.

Each of these can depend on coordinates

$$\hat{e}_i = \hat{e}_i(q_1, q_2, q_3), \quad h_k = h_k(q_1, q_2, q_3)$$

Cartesian Coordinates: Euclidean 3-dimensional "flat" space $\mathbb{R}^3$

$$\{q_1, q_2, q_3\} = \{x, y, z\} = \{x_1, x_2, x_3\} = \{r_1, r_2, r_3\}$$

unit vectors: $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\} = \{\hat{i}, \hat{j}, \hat{k}\} = \{\hat{x}, \hat{y}, \hat{z}\}$.

$\cancel{\hat{i}(x,y,z)}$ $\hat{i}$ is a constant vector.

Scale functions

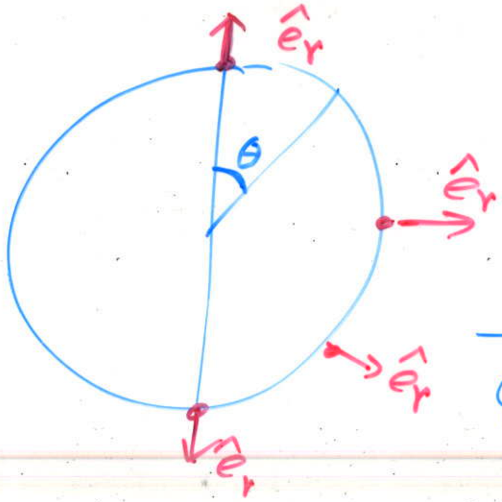
$$\{h_x = 1, \ h_y = 1, \ h_z = 1\}$$

e.g. volume integral $= \iiint dV = \iiint (dx\, dy\, dz)$

$= \int h_x\, dx \int h_y\, dy \int h_z\, dz$

Spherical Polar Coordinates: $\{q_1, q_2, q_3\} = \{r, \theta, \varphi\}$

Physics definitions



θ = polar angle

$\theta = 0 \Rightarrow$ North pole

$\theta = \pi \Rightarrow$ South pole

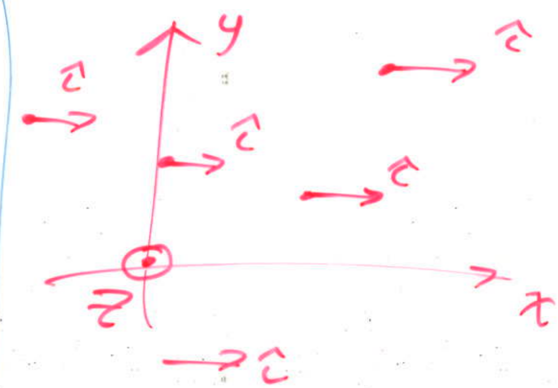
φ = azimuthal angle

$0 \le \varphi \le 2\pi$

Unit vectors: $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$

$\hat{e}_r = \hat{e}_r(\theta, \varphi)$

In contrast



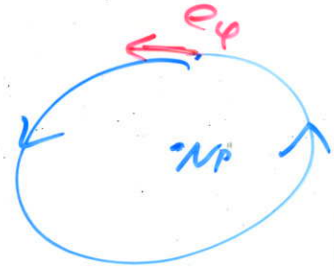
On earth

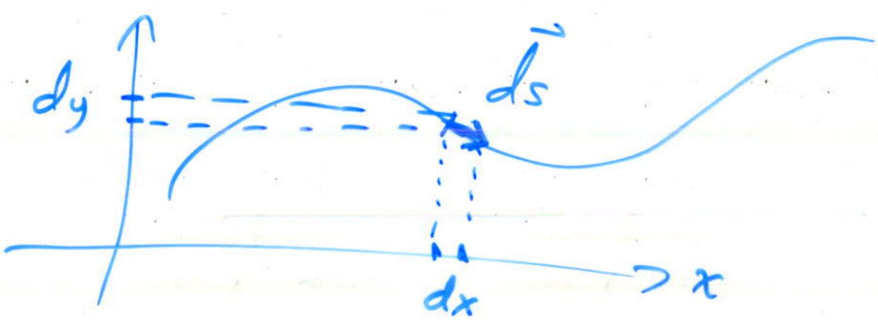
$\hat{e}_r$ = up

$\hat{e}_\theta$ = south

$\hat{e}_\varphi$ = east

right-handed system





$$\vec{ds} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k} = \sum_{i=1}^{3} dx_i\,\hat{e}_i$$

Scale functions $\{h_r, h_\theta, h_\varphi\}$

Consider (line element)^2

$$\text{line element}^2 = ds^2 = \vec{ds}\cdot\vec{ds} = dx^2 + dy^2 + dz^2$$

$$\text{line element } |d\vec{s}| = \sqrt{dx^2 + dy^2 + dz^2}$$

$$x = r\sin\theta\cos\varphi \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$y = r\sin\theta\sin\varphi \qquad \theta = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right) \pm \pi$$

$$z = r\cos\theta \qquad \varphi = \arctan\left(\frac{y}{x}\right) \pm \pi$$

$$dx = dr\sin\theta\cos\varphi + r\cos\theta\, d\theta\cos\varphi +- r\sin\theta\sin\varphi\, d\varphi$$

$$dy = dr\sin\theta\sin\varphi + r\cos\theta\, d\theta\sin\varphi + r\sin\theta\cos\varphi\, d\varphi$$

$$dz = dr\cos\theta - r\sin\theta\, d\theta$$

$$dx^2 + dy^2 + dz^2 = ds^2 = \underset{h_r^2}{1}\, dr^2 + \underset{h_\theta^2}{r^2}\, d\theta^2 + \underbrace{r^2\sin^2\theta}_{h_\varphi^2}\, d\varphi^2$$

$h_r = 1$

$h_\theta = r$

$h_\varphi = r\sin\theta$

$$\iiint dV = \iiint h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$$

$$= \iiint r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Cylindrical Polar Coordinates

$$\{q_1, q_2, q_3\} = \{s, \varphi, z\} = \{\rho, \varphi, z\} = \cancel{\{r, \varphi, z\}}$$

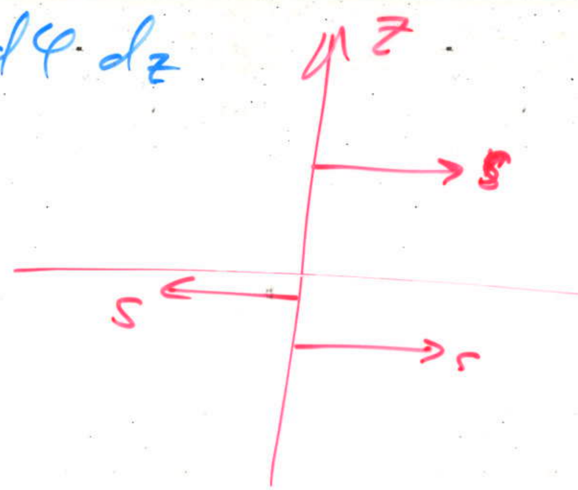
same as spherical polar φ

same as Cartesian z-coordinate.

unit vectors $\{\hat{e}_s, \hat{e}_\varphi, \hat{e}_z\}$

scale functions: $h_s = 1$, $h_\varphi = s$, $h_z = 1$

$$\iiint dV = \iiint s \, ds \, d\varphi \, dz$$



$$s = \sqrt{x^2 + y^2}$$

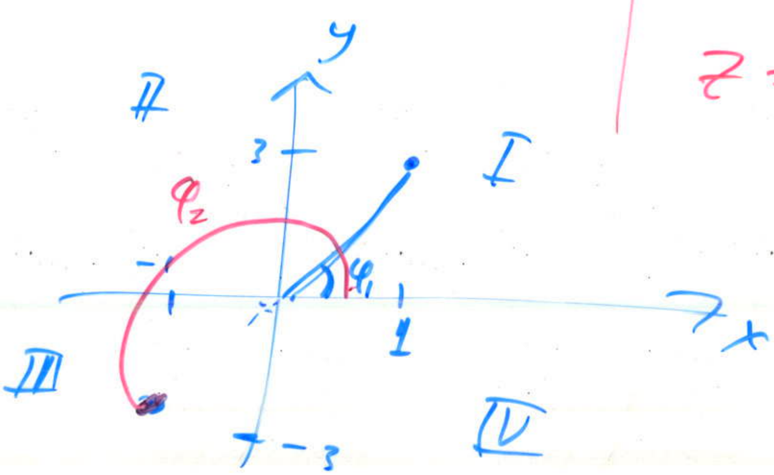
$$\varphi = \arctan\left(\frac{y}{x}\right) \pm \pi$$

$$z = z$$

$$x = s \cos\varphi$$

$$y = s \sin\varphi$$

$$z = z$$



$$\varphi_1 = \arctan\left(\frac{3}{1}\right) =$$

$$\varphi_2 = \arctan\left(\frac{-3}{-1}\right) = \arctan\left(\frac{3}{1}\right)$$