Now with functions

Key concept: Hilbert Space

Normed vector space, possibly ∞ dimensional

- Functional Analog of the dot product.
- A set of Basis functions $\{u_1(t), u_2(t), \dots\}$

$$\hat{u}_i \cdot \hat{u}_j \longleftrightarrow \frac{1}{N}\int_{t_1}^{t_2} u_i^*(t)\, u_j(t)\, dt = \langle u_i(t) | u_j(t) \rangle$$

$$= \delta_{ij}$$

e.g. Taylor basis set of functions

$$\{1, x, x^2, x^3, \dots\}$$

e.g. Hermite polynomials, Laguerre, Laplace, ...

e.g. Fourier basis set of functions

$\cos(0\omega t)$

$$\left\{ \begin{matrix} 1, & \cos(\omega t), & \cos(2\omega t), & \cos(3\omega t), & \dots \\ & \sin(\omega t), & \sin(2\omega t), & \sin(3\omega t), & \dots \end{matrix} \right\}$$

How to chose ω, the fundamental frequency?

$\omega = \frac{2\pi}{T}$ where T is the period.

This idea will only work if

a) the function $F(t)$ is periodic

b) the function $F(t)$ is only defined on a finite t interval. Then we can <u>make</u> $F(t)$ periodic.

If $F(t)$ is defined on $-\infty < t < +\infty$ and $F(t)$ is <u>not</u> periodic, then use Fourier ~~series~~ transform.

$$\langle 1 | 1 \rangle = \frac{1}{N} \int_{t_1}^{t_2} 1 \cdot 1 \cdot dt = \frac{2}{T} \int_0^T dt = \frac{2}{T}(T-0) = \boxed{2}$$

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integer number of periods $T = \frac{2\pi}{\omega}$

$$\langle \cos(\omega t) | \cos(\omega t) \rangle = \frac{1}{N} \int_{t=0}^{T} \cos^2(\omega t)\, dt =$$

$$= \frac{2}{T} \int_{-T/2}^{+T/2} \cos^2(\omega t)\, dt = \boxed{1}$$

$$\underbrace{\frac{1}{N} = \frac{2}{T}}$$

for all $\cos(n\omega t)$

$\sin(n\omega t)$ → use it for $|1\rangle$ also.

$$\frac{2}{T} \int_0^T \cos^2(\omega t)\, dt = 2\left[\cos^2(\omega t)\right]_{AVG} = 2\left[\tfrac{1}{2}\right] = \boxed{1}$$

$$\langle 1 | \cos(\omega t) \rangle = \frac{2}{T} \int_0^T 1 \cdot \cos(\omega t)\, dt = \boxed{0}$$

$$\langle 1 \mid \sin(\omega t)\rangle = \frac{2}{T}\int_0^T 1\cdot\sin(\omega t)\,dt = 0$$

$$\langle \sin(\omega t) \mid \sin(\omega t)\rangle = \frac{2}{T}\int_0^T \sin^2(\omega t)\,dt = 2\left[\tfrac{1}{2}\right] = \boxed{1}$$

$$\langle \sin(\omega t) \mid \cos(\omega t)\rangle = \frac{2}{T}\int_0^T \sin(\omega t)\cos(\omega t)\,dt = \boxed{0}$$

$$\langle \cos(n\omega t) \mid \cos(p\omega t)\rangle = \frac{2}{T}\int_0^T \cos(n\omega t)\cos(p\omega t)\,dt = \delta_{np}$$

$$\langle \sin(n\omega t) \mid \sin(p\omega t)\rangle = \text{- - - - -} = \delta_{np}$$

$$\langle \cos(n\omega t) \mid \sin(p\omega t)\rangle = 0$$

Last time: $\vec{F} = \sum_{n=1}^{3} b_n \hat{u}_n$ ← basis vector

↖ coefficients

Now: $F(t) = \frac{a_0}{2}1 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$

coefficients

basis functions

Last time: $b_n = \hat{u}_n \cdot \vec{F}$

$a_n = \langle \cos(n\omega t) \mid F(t)\rangle$ for $n = 0, 1, 2, 3\ldots$

$b_n = \langle \sin(n\omega t) \mid F(t)\rangle$ for $n = 1, 2, 3\ldots$

$$\langle 1 | F(t) \rangle = \left\langle 1 \middle| \frac{a_0}{2} 1 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t) \right] \right\rangle$$

$$= \frac{a_0}{2} \underbrace{\langle 1 | 1 \rangle}_{2} + \sum_{n=1}^{\infty} \Big[a_n \cancelto{0}{\langle 1 | \cos(n\omega t) \rangle}$$

$$+ b_n \cancelto{0}{\langle 1 | \sin(n\omega t) \rangle} \Big]$$

$$a_0 = \langle 1 | F(t) \rangle = \frac{2}{T} \int_0^T F(t)\, dt = 2 (F)_{AVG}$$

$p \neq 0$

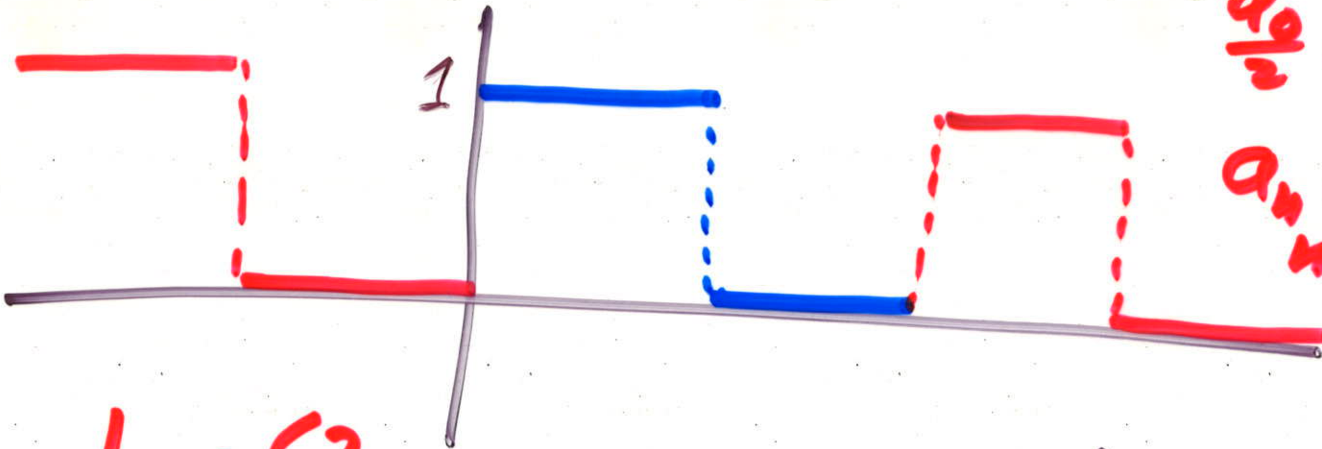
$$\langle \cos(p\omega t) | F(t) \rangle = \left\langle \cos(p\omega t) \middle| \frac{a_0}{2} 1 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t) \right] \right\rangle$$

$$= \frac{a_0}{2} \cancelto{0}{\langle \cos(p\omega t) | 1 \rangle} + \sum_{n=1}^{\infty} a_n \underbrace{\langle \cos(p\omega t) | \cos(n\omega t) \rangle}_{\delta_{np}}$$

$$+ \sum_{n=1}^{\infty} b_n \cancelto{0}{\langle \cos(p\omega t) | \sin(n\omega t) \rangle}$$

$$\langle \cos(p\omega t) | F(t) \rangle = \sum_{n=1}^{\infty} a_n \delta_{np} = a_p$$

$$\langle \sin(p\omega t) | F(t) \rangle = \sum_{n=1}^{\infty} b_n \delta_{np} = b_p$$



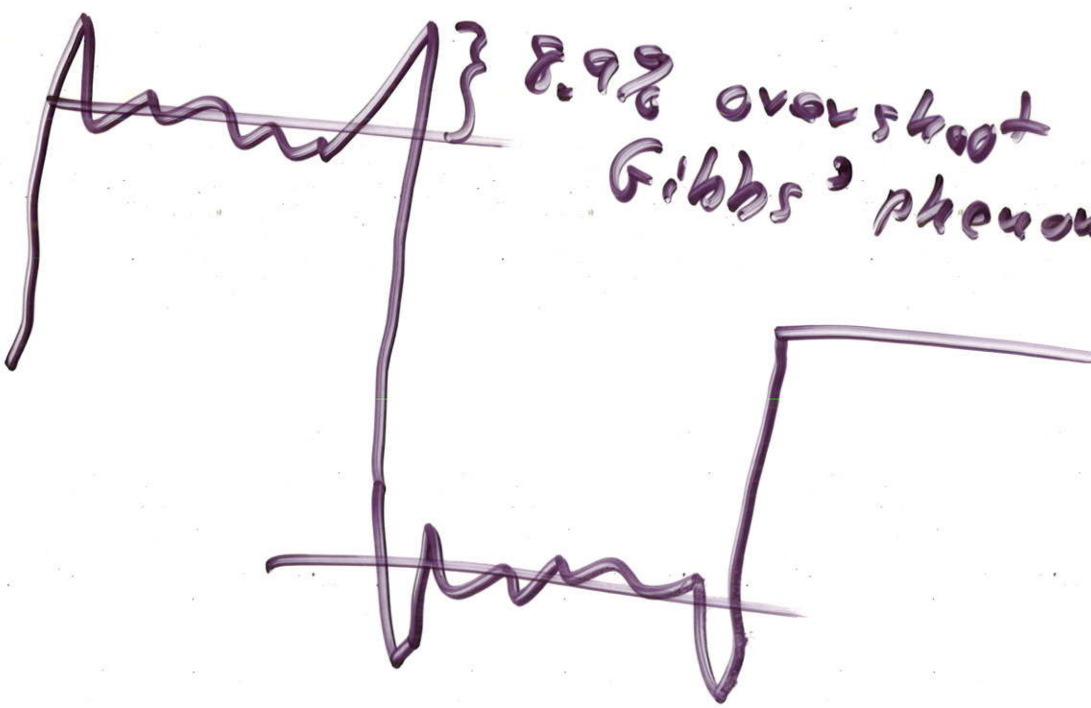
$\frac{a_0}{2} = \frac{1}{2}$

$a_n = 0$
$n \geq 1$

$$b_n = \begin{cases} \frac{2}{\pi n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

discontinuity
$\rightarrow b_n \propto \frac{1}{n}$

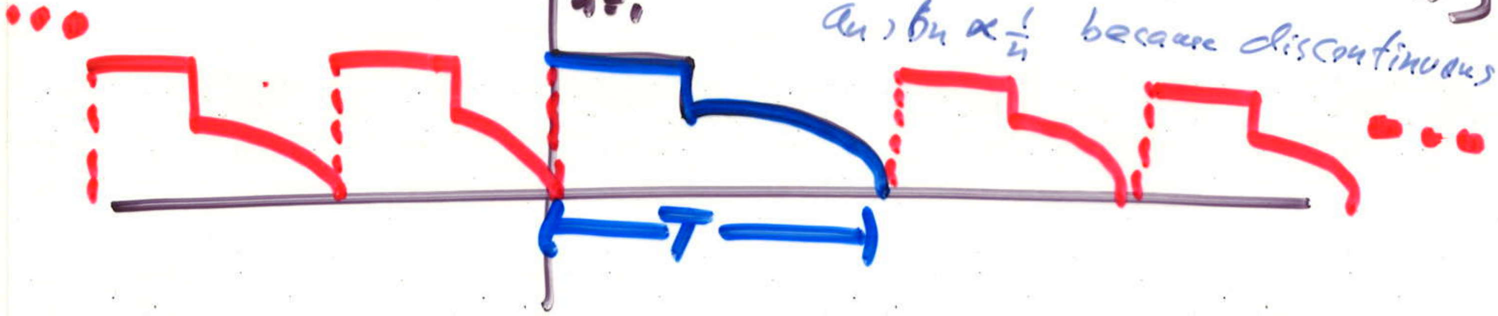
"almost odd" – would be odd if shifted vertically $\frac{1}{2}$ unit down



8.9% overshoot
Gibbs' phenomenon

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t) \right]$$

$a_n, b_n \propto \frac{1}{n}$ because discontinuous

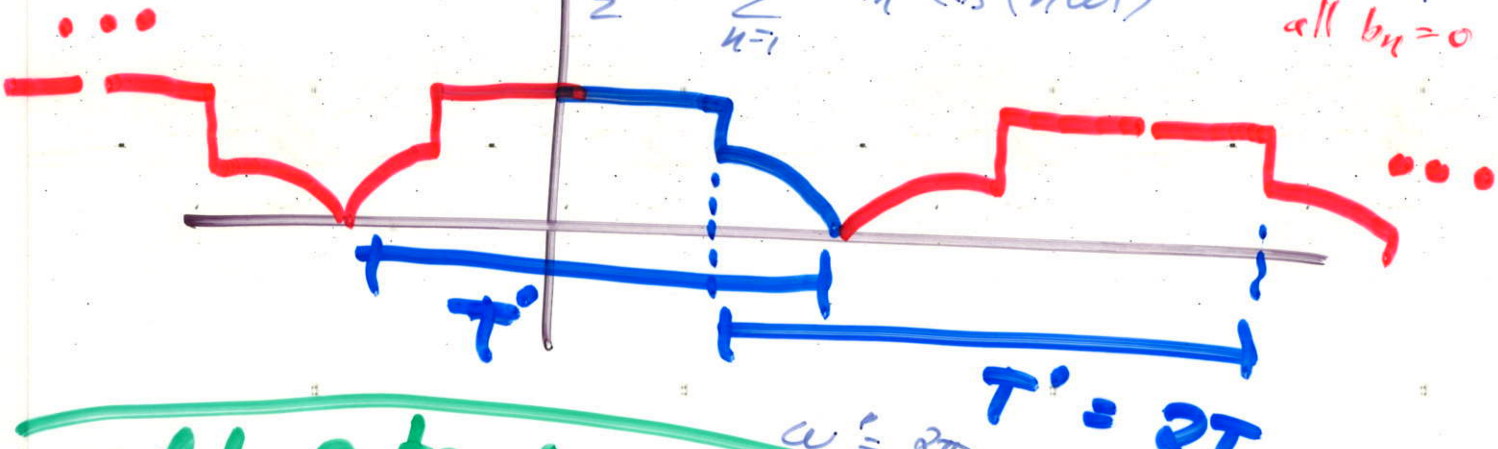


even extension

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega' t)$$

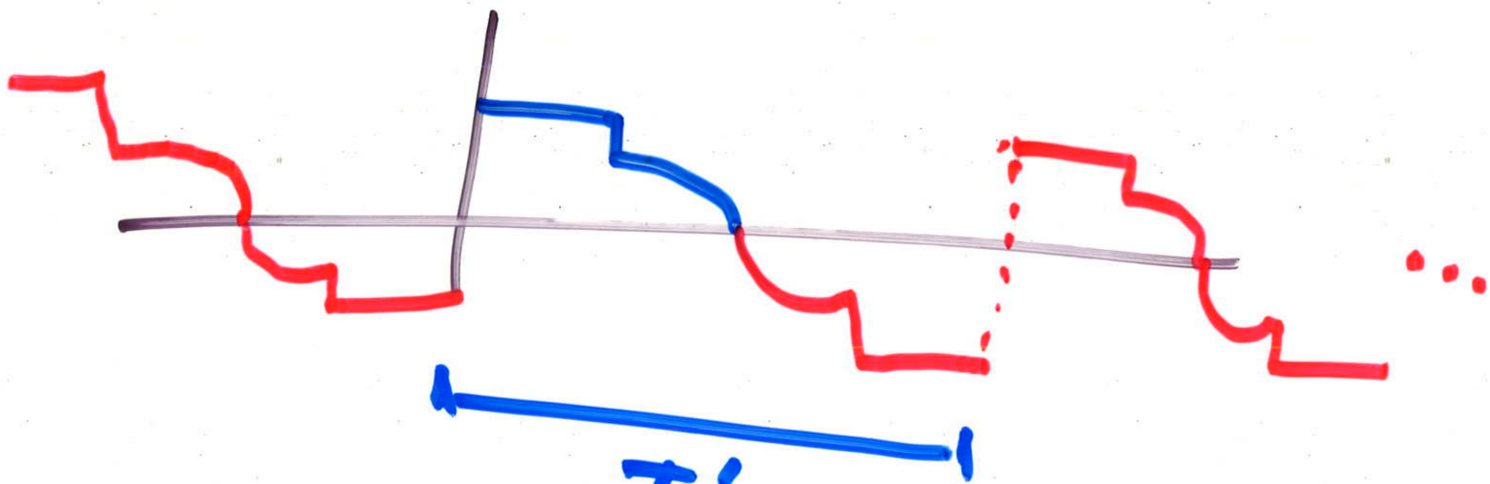
continuous $a_n, b_n \propto \frac{1}{n^2}$

all $b_n = 0$



$\omega' = \frac{2\pi}{T'}$

odd extension

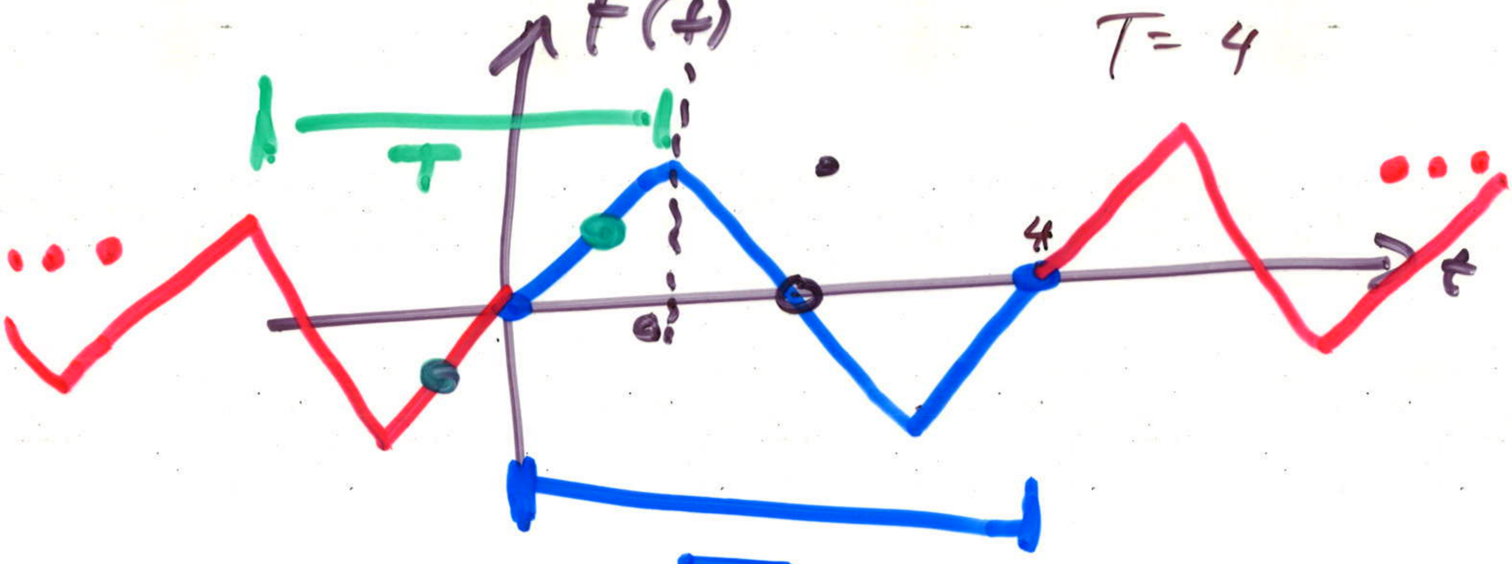


$$F(t) = \sum_{n=1}^{\infty} b_n \sin(n\omega' t)$$

— all $a_n = 0$

$\uparrow$

$\propto \frac{1}{n}$



T period $= \frac{2\pi}{\omega}$

Odd: $F(-t) = -F(t)$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{4}$

even $F(-t) = +F(t)$

verify

$b_1 = \frac{8}{\pi^2}$

$b_2 = 0$

$b_3 = -\frac{8}{9\pi^2}$

$b_4 = 0$

$b_5 = \frac{8}{25\pi^2}$

$b_6 = 0$

$\vdots$

$$b_n = \frac{8}{\pi^2 n^2}(-1)^{\frac{n-1}{2}}\left[\frac{1+(-1)^{n+1}}{2}\right]$$

Great convergence.

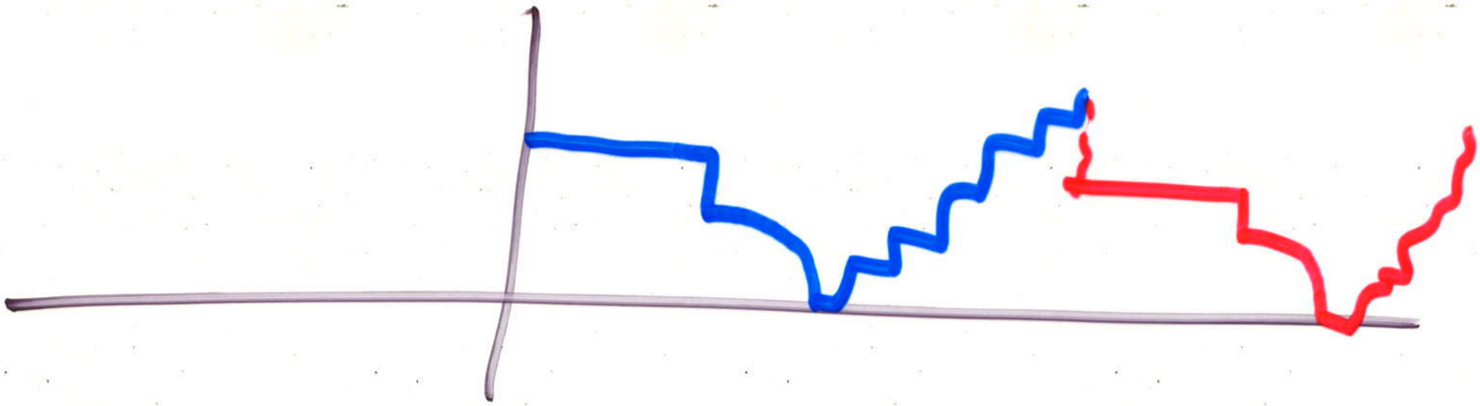
$a_0 = 0$

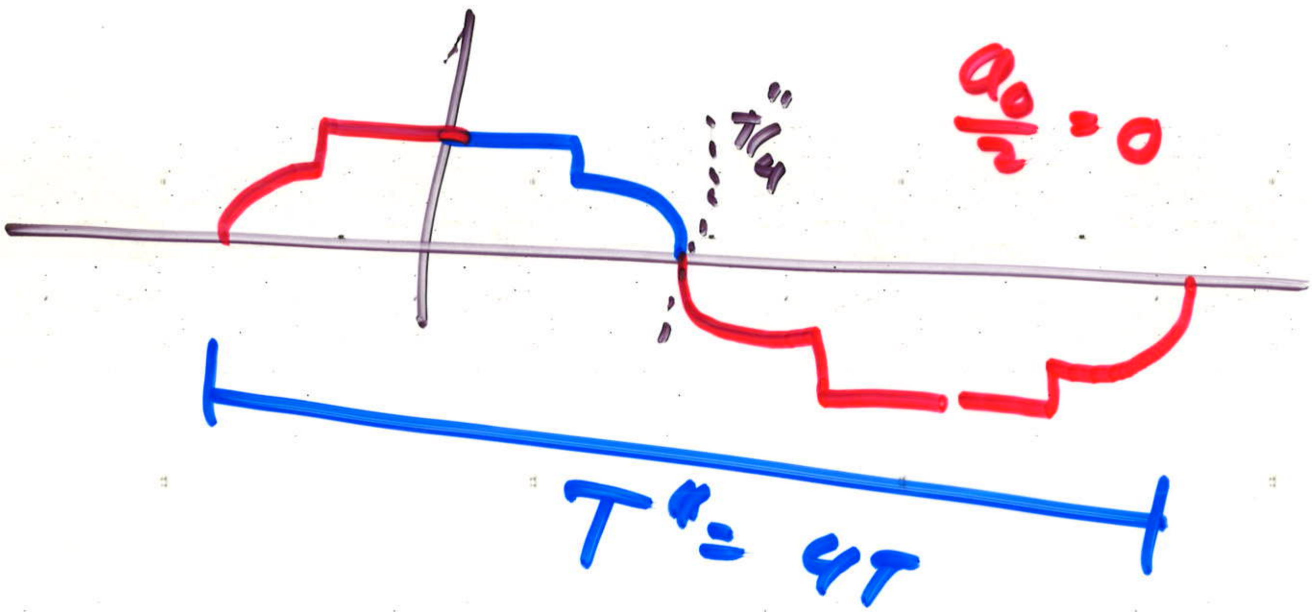
$a_n = 0 \quad \forall n$

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$F(t) = b_1 \sin(\omega t) + b_3 \sin(3\omega t) + b_5 \sin(5\omega t)$$

$$\doteq \frac{8}{\pi^2}\sin\left(\frac{\pi}{2}t\right) - \frac{8}{9\pi^2}\sin\left(\frac{3\pi}{2}t\right) + \frac{8}{25\pi^2}\sin\left(\frac{5\pi}{2}t\right)$$




T/4
$\frac{a_0}{2} = 0$
$T^* = 4T$