First-Order Differential Operators

gradient: $\vec{\nabla}\, T(\vec{r}) = \vec{V}(\vec{r})$

$\uparrow$ scalar $\quad$ $\uparrow$ vector

divergence: $\vec{\nabla} \cdot \vec{V}(\vec{r}) = S(\vec{r})$

dot product $\quad$ $\uparrow$ vector $\quad$ $\uparrow$ scalar

curl: $\vec{\nabla} \times \vec{V}(\vec{r}) = \vec{W}(\vec{r})$

cross product $\quad$ $\uparrow$ vector $\quad$ $\uparrow$ vector

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Second-Order Differential Operators

The gradient returns a vector field. $\Rightarrow$ div or curl.

div grad $T(\vec{r}) = \vec{\nabla} \cdot \vec{\nabla}\, T(\vec{r}) = S(\vec{r})$

also written: $\nabla^2 T(\vec{r}) = \Delta T(\vec{r})$ Laplacian

$$T(\vec{r}) \longrightarrow \boxed{\text{Laplacian}} \longrightarrow S(\vec{r})$$

Cartesian: grad $T(\vec{r}) = \vec{\nabla}\, T(x,y,z) = \hat{e}_x \frac{\partial T}{\partial x} + \hat{e}_y \frac{\partial T}{\partial y} + \hat{e}_z \frac{\partial T}{\partial z}$

$$\nabla^2 T(\vec{r}) = \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right) \cdot \left(\hat{e}_x \frac{\partial T}{\partial x} + \hat{e}_y \frac{\partial T}{\partial y} + \hat{e}_z \frac{\partial T}{\partial z}\right)$$

$$= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

(no unit vectors)

Spherical Polar Coordinates

$$\nabla^2 T(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\varphi^2}$$

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curl grad $T_{(\vec{r})}$ = $\vec{\nabla}\times\vec{\nabla}T(\vec{r}) = \vec{0} \quad \forall\, T(\vec{r})$

Proof:
$$\vec{\nabla}\times\vec{\nabla}T = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} & \frac{\partial T}{\partial z} \end{vmatrix}$$

$$= \hat{e}_x \underbrace{\left[\frac{\partial}{\partial y}\frac{\partial T}{\partial z} - \frac{\partial}{\partial z}\frac{\partial T}{\partial y}\right]}_{\text{mixed partial derivatives}} + \hat{e}_y \underbrace{[\,\cdots]}_{0} + \hat{e}_z \underbrace{[\,\cdots]}_{0}$$

$$= \vec{0} = (0,0,0)$$

The divergence returns a scalar field - only the gradient can act on it.

grad div $\vec{V}(\vec{r}) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{V}(\vec{r})\right) = \vec{U}(\vec{r})$

The curl returns a vector field - take div or curl

div curl $\vec{V}(\vec{r}) = \vec{\nabla}\cdot\left[\vec{\nabla}\times\vec{V}(\vec{r})\right] = 0 \quad \forall\, \vec{V}(\vec{r})$

$$\vec{\nabla}\times\vec{V}(\vec{r}) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ V_x & V_y & V_z \end{vmatrix}$$

$$= \hat{e}_x\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \hat{e}_y\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \hat{e}_z\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{V}(\vec{r}) = \frac{\partial}{\partial x}\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \frac{\partial}{\partial y}\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

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curl curl $\vec{V}(\vec{r}) = \vec{\nabla}\times\vec{\nabla}\times\vec{V}(\vec{r}) = \vec{A}(\vec{r})$

vector   vector

The Laplacian of $\frac{1}{r}$ is a remarkable function

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right] = \frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right]$$

$$= \frac{1}{r^2}\frac{d}{dr}\left[r^2\left(-\frac{1}{r^2}\right)\right] = \frac{1}{r^2}\frac{d}{dr}\left[-1\right] = \frac{0}{r^2}$$

$$= \begin{cases} 0 & r \neq 0 \\ ? & r = 0 \end{cases}$$

Use the Divergence Theorem

evaluate on the surface S.

$$\iiint_{\mathcal{V}} \vec{\nabla}\cdot \vec{V}(\vec{r})\, d\mathcal{V} = \oiint_{S} \vec{V}(\vec{r})\cdot d\vec{A}$$

use $\vec{\nabla}\left(\frac{1}{r}\right)$

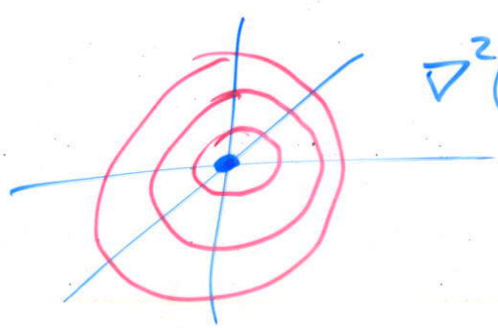
for S, choose a sphere of radius $\underline{a}$ around the origin.

$$\iiint_{\mathcal{V}} \nabla^2\left(\frac{1}{r}\right) d\mathcal{V} = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} \vec{\nabla}\left(\frac{1}{r}\right)\Big|_{r=a} \cdot \underbrace{\hat{e}_r\, \underbrace{a^2 \sin\theta}_{h_\theta \cdot h_\varphi}\, d\theta\, d\varphi}_{d\vec{A}}$$

$$\vec{\nabla}\left(\frac{1}{r}\right) = \hat{e}_r \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \hat{e}_\theta\, \emptyset + \hat{e}_\varphi\, \emptyset = -\frac{\hat{e}_r}{r^2}$$

$$= \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} -\frac{\hat{e}_r}{\cancel{a^2}} \cdot \hat{e}_r\, \cancel{a^2} \sin\theta\, d\theta\, d\varphi = -4\pi$$

no a dependence!



$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\, \delta^{(3)}(\vec{r})$$

$$= -4\pi\, \delta(x)\delta(y)\delta(z) \checkmark$$

$$= -\frac{\delta(r)}{r^2} \checkmark$$

Check! Cartesian:

$$\int_{x=-\infty}^{+\infty}\int_{y}\int_{z} -4\pi\,\delta(x)\delta(y)\delta(z)\,dx\,dy\,dz = -4\pi\int_{x=-\infty}^{+\infty}\delta(x)\,dx\int_{y=-\infty}^{+\infty}\delta(y)\,dy\int_{z=-\infty}^{+\infty}\delta(z)\,dz$$

$$= -4\pi(1)(1)(1) = -4\pi$$

Spherical Polars

$$\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\frac{-\delta(r)}{r^2}\,r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= \underbrace{-\int_{r=0}^{\infty}\delta(r)\,dr}_{-1}\ \underbrace{\int_{\theta=0}^{\pi}\sin\theta\,d\theta}_{2}\ \underbrace{\int_{\varphi=0}^{2\pi}d\varphi}_{2\pi} = -4\pi\ \checkmark$$

Gaussian Integrals

$$I = \int_{x=-\infty}^{+\infty} e^{-x^2}\,dx$$

$$I^2 = \left[\int_{x=-\infty}^{+\infty} e^{-x^2}\,dx\right]\left[\int_{y=-\infty}^{+\infty} e^{-y^2}\,dy\right]$$

$$I^2 = \int_{y=-\infty}^{+\infty}\int_{x=-\infty}^{+\infty}\underbrace{e^{-x^2}e^{-y^2}}_{e^{-(x^2+y^2)} = e^{-s^2}}\,dy\,dx$$

change variables to polar coordinates

$s = \sqrt{x^2+y^2}$; $dx\,dy = s\,ds\,d\varphi$

$\nwarrow$ 4φ

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$$I^2 = \int_{s=0}^{\infty}\int_{\varphi=0}^{2\pi} e^{-s^2} s \, ds \, d\varphi = \left[\int_{s=0}^{\infty} e^{-s^2} s \, ds\right] \underbrace{\left[\int_{\varphi=0}^{2\pi} d\varphi\right]}_{2\pi}$$

change variable $u = s^2$
$du = 2s \, ds$

$$I^2 = 2\pi \int_{u=0}^{\infty} e^{-u} \frac{1}{2} du = \pi \left[-e^{-u}\right]_{u=0}^{\infty} = \pi\left[-\underbrace{e^{-\infty}}_{0} + \underbrace{e^{-0}}_{1}\right]$$

$$I^2 = \pi \implies I = \sqrt{\pi}$$

$$\int_{y=-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\pi}$$

change-variables
$y = \sqrt{a}\, x \; ; \; y^2 = a x^2$
$dy = \sqrt{a}\, dx$

$$\int_{x=-\infty}^{+\infty} e^{-ax^2} \sqrt{a}\, dx = \sqrt{\pi}$$

$$\int_{x=-\infty}^{+\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

Differentiate under the integral sign:

$$\frac{d}{da}\left[\int_{x=-\infty}^{+\infty} e^{-ax^2}\,dx\right] = \frac{d}{da}\left[\sqrt{\frac{\pi}{a}}\right]$$ then set $a=1$

$$\parallel$$

$$-\int_{x=-\infty}^{+\infty} x^2 e^{-ax^2}\,dx\,\Bigg|_{a=1} = -\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{a^3}}\Bigg|_{a=1}$$

$$\Rightarrow \int_{x=-\infty}^{+\infty} x^2 e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$\int_{x=-\infty}^{+\infty} x^4 e^{-x^2}\,dx = \frac{3\sqrt{\pi}}{4}$$

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odd powers of x?

e.g. $$\int_{x=-\infty}^{+\infty} \underbrace{x^3}_{\text{odd}}\, \underbrace{e^{-x^2}}_{\text{even}}\,dx = ? = 0$$