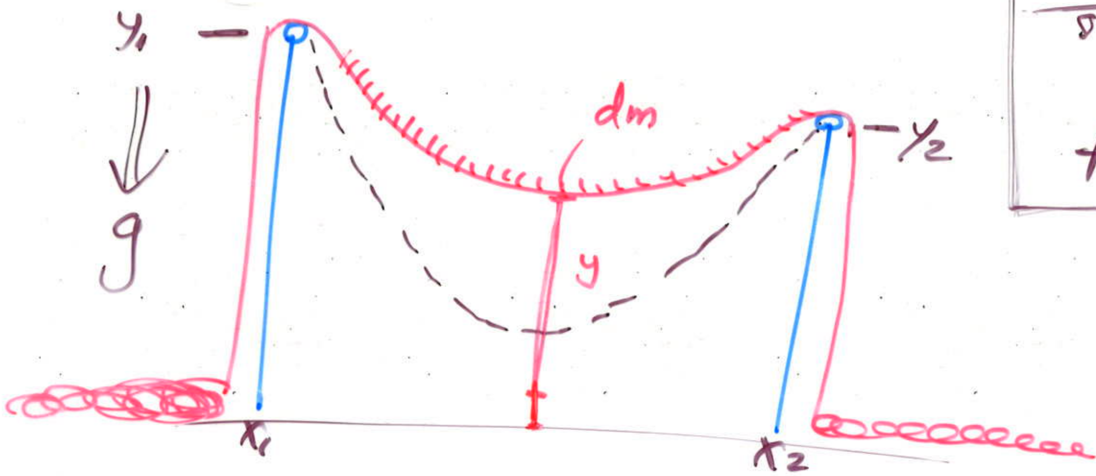
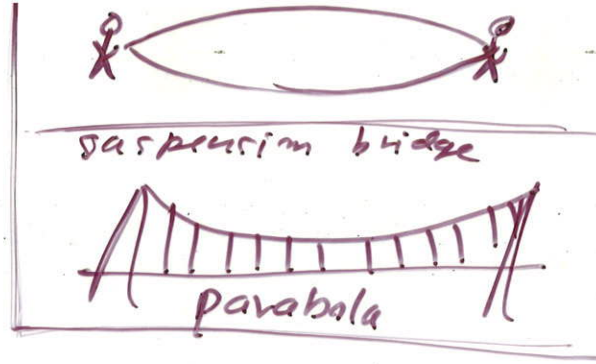
Catenary (unconstrained)





potential energy

$\leftarrow U = 0$

$y = 0$

Functional is Potential energy $\quad dm = \mu\, ds \leftarrow$ arc length

μ ↑ linear mass density

$$U = \int g\, y\, dm = \int g\, y\, \mu\, ds$$

$$U = g\mu \int_{x_1}^{x_2} y\sqrt{1+y'^2}\,dx$$

$$G = y\sqrt{1+y'^2} = y\,(1+y'^2)^{1/2}$$

$$\frac{\partial G}{\partial y} - \frac{d}{dx}\left[\frac{\partial G}{\partial y'}\right] = 0 \qquad \text{Euler}$$

$$\frac{\partial G}{\partial y} = \sqrt{1+y'^2} \quad // \quad \frac{\partial G}{\partial y'} = \frac{y\,y'}{\sqrt{1+y'^2}}$$

$$\sqrt{1+y'^2} - \frac{d}{dx}\left\{\frac{y\,y'}{\sqrt{1+y'^2}}\right\} = 0$$

2nd order D.E. for $y(x)$

Try Beltrami : $\frac{\partial G}{\partial x} = 0$

$$G - y' \frac{\partial G}{\partial y'} = \text{constant} \equiv a$$

$$y\sqrt{1+y'^2} - y' \frac{yy'}{\sqrt{1+y'^2}} = a$$

$$\frac{y(1+y'^2) - yy'^2}{\sqrt{1+y'^2}} = a = \frac{y}{\sqrt{1+y'^2}}$$ square both sides

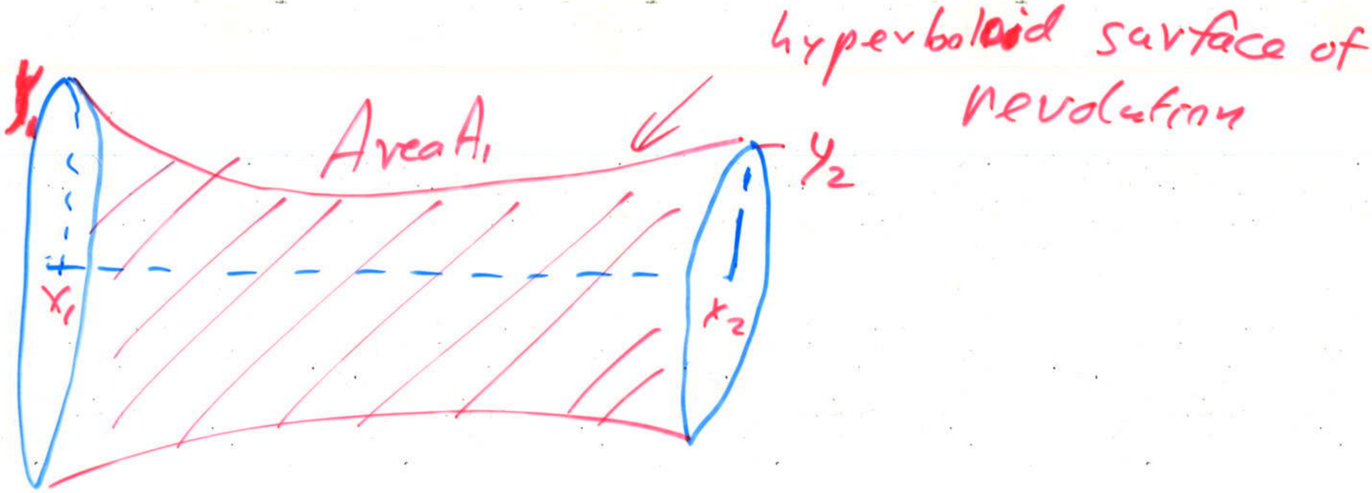
$$\frac{y^2}{1+y'^2} = a^2 \quad \text{solve for } y' \implies \frac{dy}{dx} = y' = \sqrt{\frac{y^2-a^2}{a^2}}$$

$$\int dx = \int \frac{a\,dy}{\sqrt{y^2-a^2}}$$

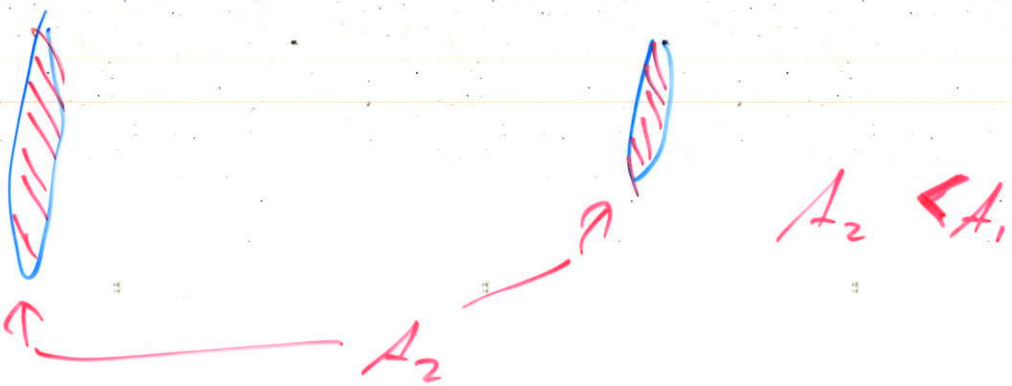
$$x = a \operatorname{arccosh}\left(\frac{y}{a}\right) + b$$

$$y(x) = a\cosh\left(\frac{x-b}{a}\right)$$

Use $\underline{a}$ and $\underline{b}$ to fit the curve through the endpoints



Goldschmidt solution



Galileo in <u>The Two Sciences</u> mentions a parabolic approximation to the catenary.

Joachim Jungius proved the curve was not a parabola in 1669 (after his death).

Math $I = \int_{x=x_1}^{x_2} G[f(x), f'(x); x]\, dx$

Euler Equation: $\delta I = 0 \implies \frac{\partial G}{\partial f} - \frac{d}{dx}\left[\frac{\partial G}{\partial f'}\right] = 0$

Physics

generalized coordinate

gen. velocity

$$S = \int_{t=t_1}^{t_2} L[q(t), \dot{q}(t); t]\, dt$$

action (functional)

Lagrangian

Euler-Lagrange: $\delta S = 0 \implies \frac{\partial L}{\partial q} - \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] = 0$

$V[q(t)]$

To recover Newton's 2nd law, $L = T - V$

kinetic energy

potential energy

$T[\dot{q}(t)]$

e.g. point mass moving in one dimension under a force. $F = -\frac{\partial V}{\partial r}$

$T = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$ $\quad V(x)$

function

$x(t)$ ← variable

$\dot{x}(t)$

$$L = T - V = \frac{m}{2}\dot{x}^2 - V(x)$$

$$S = \int_{t=t_1}^{t_2} L\, dt = \int_{t=t_1}^{t_2} \left[\frac{m}{2}\dot{x}^2 - V(x)\right] dt$$

Euler-Lagrange Equation

$$\delta S = 0 \Rightarrow \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \Rightarrow F_x - m a_x = 0$$

$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial x} = -\frac{dV(x)}{dx} = F_x$$

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad / \quad \frac{d}{dt}(m\dot{x}) = m\ddot{x} = m a_x$$

Suppose L depends on more than 1 gen. coordinate.

e.g. $T = \frac{1}{2} m v^2 = \frac{m}{2}\left(V_x^2 + V_y^2\right) = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$

$V(x, y)$

$$L = T - V = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - V(x, y)$$

$$S = \int_{t=t_1}^{t_2} \left[\frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - V(x,y)\right] dt$$

<u>Two</u> Euler-Lagrange Equations.

$$\delta S = 0 \Rightarrow \begin{cases} \dfrac{\partial L}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{x}}\right) = 0 \Rightarrow \overset{-\frac{\partial V}{\partial x}}{F_x} = m\overset{\ddot{x}}{a_x} \\[2ex] \dfrac{\partial L}{\partial y} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{y}}\right) = 0 \Rightarrow \underset{-\frac{\partial V}{\partial y}}{F_y} = m\underset{\ddot{y}}{a_y} \end{cases}$$

Suppose the Lagrangian L depends on N gen. coordinates $\{q_i\}$ $i = 1, 2 \ldots N$.

$q_1(t), \dot{q}_1(t), q_2(t) \ldots$

$$L = T(\dot{q}_1, \dot{q}_2, \ldots \dot{q}_N) - V(q_1, q_2, \ldots q_N)$$

$\delta S = 0 \Rightarrow N$ Euler-Lagrange Equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \quad \forall i \quad i = 1, \ldots N$$

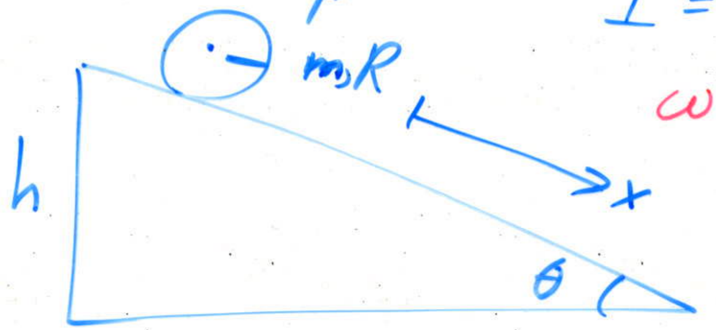
Generalized coordinates need not be lengths

e.g. $\{q_i\} = \{r, \theta, \varphi\}$ $r(t), \theta(t), \varphi(t)$

Length (r) angles (θ, φ)

$\{q_i\}$ could be speeds, energies, Fourier Coefficients...

e.g. Solid cylinder rolling without slipping down a ramp.



$I = \frac{1}{2} m R^2$ moment of inertia about the center of mass.

$\omega = \frac{v_{cm}}{R}$

$$T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \omega^2$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2}\left(\frac{1}{2} m \cancel{R^2}\right)\frac{\dot{x}^2}{\cancel{R^2}}$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{4} m \dot{x}^2 = \frac{3}{4} m \dot{x}^2$$

$$V = -m g x \sin\theta + \underline{\text{const}}$$

$$L = T - V = \frac{3}{4} m \dot{x}^2 + m g x \sin\theta$$

$$\frac{\partial L}{\partial x} = m g \sin\theta \qquad \frac{\partial L}{\partial \dot{x}} = \frac{3}{2} m \dot{x}$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \implies m g \sin\theta - \frac{d}{dt}\left(\frac{3}{2} m \dot{x}\right) = 0$$

$$\cancel{m} g \sin\theta - \frac{3}{2} \cancel{m} \ddot{x} = 0$$

$$\ddot{x} = \boxed{a_{\text{rolling}} = \frac{2}{3} g \sin\theta} < a_{\text{sliding}} = g \sin\theta$$

Could redo this problem using φ as the generalized coordinates