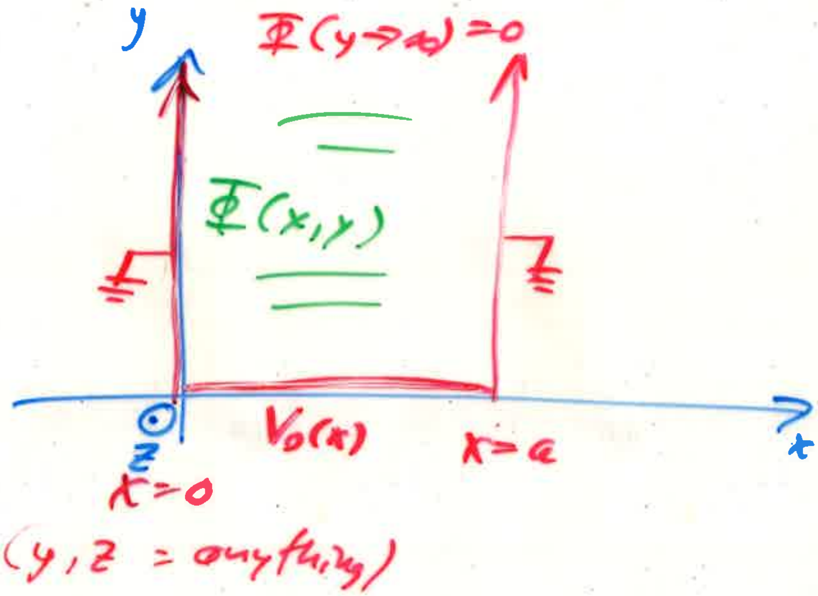
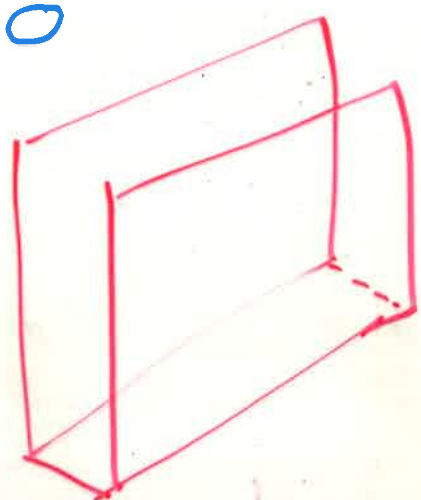
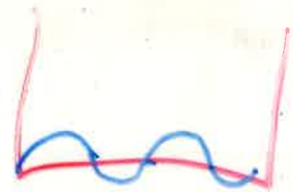
Separation of Variables

B.C. depend on x, y

Laplace Eq. $\nabla^2 \Phi(\vec{r}) = 0$







Ansatz: $\Phi(x,y) = F(x)\,G(y)$

2nd order linear PDE

$$\nabla^2 \Phi(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left[F(x)\,G(y)\right] = 0$$

$$= \frac{d^2 F(x)}{dx^2}\,G(y) + F(x)\,\frac{d^2 G(y)}{dy^2} = 0$$

$$\underbrace{\frac{\frac{d^2F(x)}{dx^2}}{F(x)}} + \underbrace{\frac{\frac{d^2G(y)}{dy^2}}{G(y)}} = 0$$

$$H(x) + J(y) = 0 \quad \forall\, x, y$$

$$-\alpha^2 + \alpha^2 = 0$$

Two 2^{nd}-order linear ODEs.

$$\frac{\frac{d^2F(x)}{dx^2}}{F(x)} = -\alpha^2 \Rightarrow \frac{d^2F(x)}{dx^2} = -\alpha^2 F(x)$$

$$\frac{d^2F(x)}{dx^2} + \alpha^2 F(x) = 0$$

$$c_3 e^{i\alpha x} + c_4 e^{-i\alpha x}$$

$$c_5 \sin(\alpha x + c_6)$$

$$c_7 \cos(\alpha x + c_8)$$

$$\rightarrow F(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x)$$

these are complete

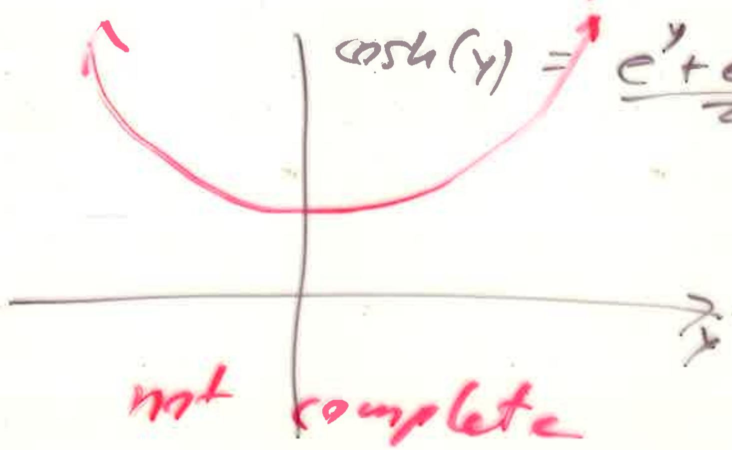
$$\frac{d^2G(y)}{dy^2} = +\alpha^2 G(y)$$

$$\frac{d^2G(y)}{dy^2} - \alpha^2 G(y) = 0$$

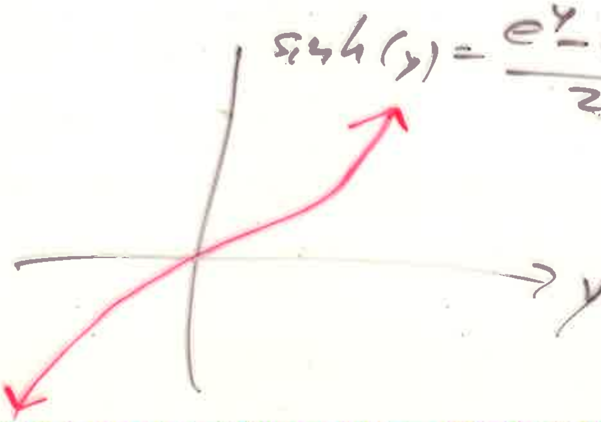
$$G(y) = \underbrace{d_1 e^{-\alpha y}}_{\text{decay}} + \underbrace{d_2 e^{+\alpha y}}_{\text{growth}}$$

also $d_3 \sinh(\alpha y) + d_4 \cosh(\alpha y)$

$$\cosh(y) = \frac{e^y + e^{-y}}{2}$$



$$\sinh(y) = \frac{e^y - e^{-y}}{2}$$



not complete

$$F(x) = C_1 \sin(\alpha x) + C_2 \cos(\alpha x)$$

① B.C. $\Phi(0,y) = 0 \Rightarrow F(0) = 0$

$$F(0) = C_1 \cancel{\sin(\alpha 0)}^{\to 0} + C_2 \cancel{\cos(\alpha 0)}^{\to 1} = \boxed{C_2 = 0}$$

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② B.C. $\Phi(a,y) = 0 \Rightarrow F(a) = 0$

$$F(x) = C_1 \sin(\alpha x)$$

$$F(a) = C_1 \sin(\alpha a) = 0 \qquad C_1 \neq 0$$

$$\Rightarrow \sin(\alpha a) = 0 \Rightarrow \alpha a = n\pi \quad , n = 1, 2, 3, \ldots$$

Separation constant α is quantized

$$\alpha = \frac{n\pi}{a} \qquad n = 1, 2, 3, \ldots$$

$$G(y) = d_1 e^{-\alpha y} + d_2 e^{+\alpha y}$$

$$= d_1 e^{-\frac{n\pi y}{a}} + d_2 e^{+\frac{n\pi y}{a}}$$

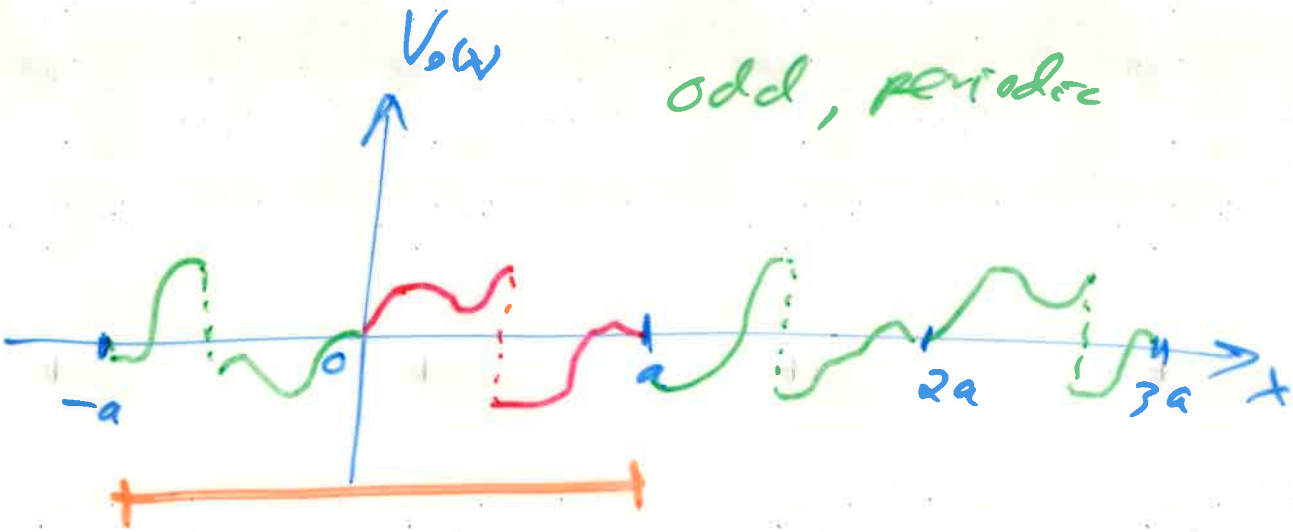
B.C. $y \to \infty$, $G(y) \to 0 \Rightarrow d_2 = 0$

$$\Phi(x,y) = F(x)\, G(y) = C_1 \sin\left(\frac{n\pi x}{a}\right) d_1 e^{-\frac{n\pi y}{a}}$$

$$\Phi(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) e^{-\frac{n\pi y}{a}}$$

One last Boundary Cond. $y = 0$, $\Phi(x,0) = V_0(x)$

$$\Phi(x,0) = V_0(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)$$



T = period
= 2a

→ Use Fourier Analysis to solve for A_n

HW