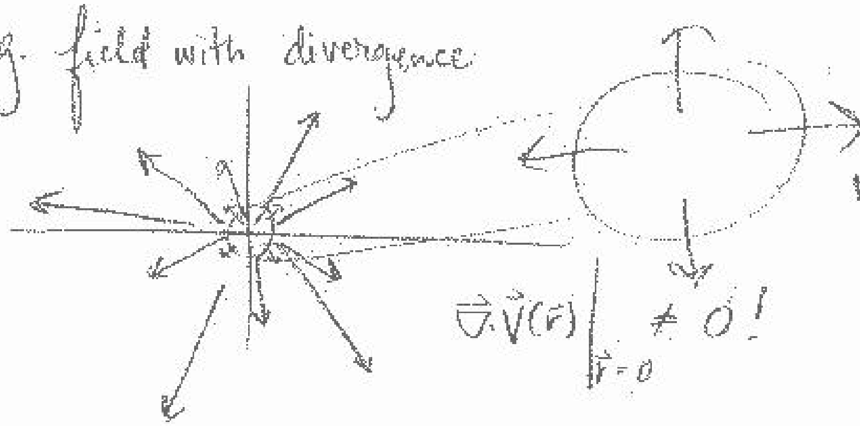
whatever goes in comes out (no sources, no sinks)

@ pt. r, everything that comes in goes out, so that $\vec{\nabla}\cdot\vec{V}(\vec{r})=0$

eg. field with divergence



$\vec{\nabla}\cdot\vec{V}(\vec{r})\Big|_{\vec{r}=0} \neq 0!$

net flow out (source)

Divergence

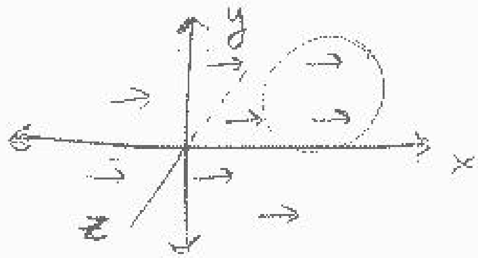
$$\vec{\nabla} \cdot \vec{V}(\vec{r})$$

in Cartesian: $\vec{\nabla} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}$

scalar

$$\vec{\nabla} \cdot \vec{V}(\vec{r}) = \frac{\partial}{\partial x} V_x(x,y,z) + \frac{\partial}{\partial y} V_y(x,y,z) + \frac{\partial}{\partial z} V_z(x,y,z)$$

where: $\vec{V} = \hat{e}_x V_x(x,y,z) + \hat{e}_y V_y(x,y,z) + \hat{e}_z V_z(x,y,z)$

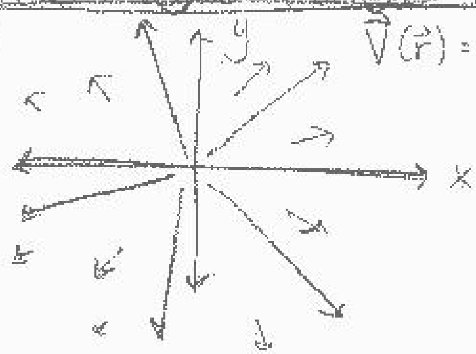


$$\vec{V}(\vec{r}) = 3\hat{e}_x = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

* any pt you pick: length of 3 vectors in x-direction

$$\vec{\nabla} \cdot \vec{V}(\vec{r}) = \frac{\partial 3}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial 0}{\partial z} = 0 \checkmark$$

b/c it looked like it should've been zero because everything that goes into any random circle you pick, also comes out (!!)



$$\vec{V}(\vec{r}) = \frac{1}{r}\hat{e}_r$$

* the farther out you get, the smaller the magnitudes of arrows become...

$$\hat{e}_r = \frac{\vec{r}}{r} = \frac{x\hat{e}_x + y\hat{e}_y + z\hat{e}_z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\vec{V}(\vec{r}) = \frac{x\hat{e}_x + y\hat{e}_y + z\hat{e}_z}{x^2+y^2+z^2}$$

$$V_x = \frac{x}{x^2+y^2+z^2}$$

$$\vec{\nabla} \cdot \vec{V}(\vec{r}) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$\vec{V}(q_1, q_2, q_3)$ for generalized coordinates

$$\vec{V}(q_1, q_2, q_3) = \hat{e}_1 V_1(q_1, q_2, q_3) + \hat{e}_2 V_2(q_1, q_2, q_3) + \hat{e}_3 V_3(q_1, q_2, \ldots$$

everything depends on (q_1, q_2, q_3)

$$V_1(q_1, q_2, q_3) = \hat{e}_1(q_1, q_2, q_3) \cdot \vec{V}(q_1, q_2, q_3)$$

Divergence, in general: *even h_i depends on $(q_1, q_2 \,\&\, q_3)$

$$\vec{\nabla} \cdot \vec{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_1 h_3) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right]$$

We want to do this in spherical polar coordinates

$h_r = 1, \; h_\theta = r, \; h_\varphi = r\sin\theta$

$$\vec{\nabla} \cdot \vec{V}(q_1, q_2, q_3) = \frac{1}{r^2\sin\theta}\Big[\frac{\partial}{\partial r}(V_r \cdot r \cdot r\sin\theta) +$$

($\partial r \to q_1$, $r \to h_2$, $r\sin\theta \to h_3$)

$$\frac{\partial}{\partial\theta}(V_\theta \cdot 1 \cdot r\sin\theta) + \frac{\partial}{\partial\varphi}(V_\varphi \cdot 1 \cdot r)\Big]$$

($\partial\theta \to q_2$, $1 \to h_1$, $r\sin\theta \to h_3$; $\partial\varphi \to q_3$, $1 \to h_1$, $r \to h_2$)

($\sin\theta$ came out b/c h_3 not dependent on r)

*
$$\boxed{\vec{\nabla} \cdot \vec{V}(q_1, q_2, q_3) = \frac{1}{r^2}\frac{\partial}{\partial r}(V_r r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(V_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}}$$

& now use it...

$\vec{V}(\vec{r}) = \frac{1}{r}\hat{e}_r + 0\hat{e}_\theta + 0\hat{e}_\phi$ & find divergence

$$\vec{\nabla}\cdot\vec{V}(\vec{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[V_r r^2\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot r^2\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r\right] = \frac{1}{r^2}$$

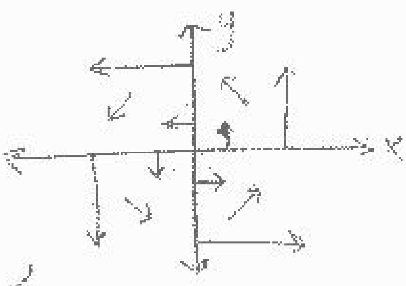
(in general ~ now evaluate it at a point)

$$\vec{\nabla}\cdot\vec{V}(\vec{r})\Big|_{\vec{r}(1,2,3)}$$

* wherever you draw the circle, more arrows will be coming out than going in...

Curl $\vec{\nabla}\times$ ___

$\vec{V} = -y\hat{e}_x + x\hat{e}_y$



four-fold symmetry

European Names

$\vec{\nabla}\cdot\vec{V} = \text{div}\,\vec{V}$ & $\vec{\nabla}f = \text{grad}(f)$

$\vec{\nabla}\times\vec{V} = \text{curl}(\vec{V}) = \text{rot}(\vec{V})$

$V_x = -y \qquad V_y = x \qquad V_z = 0$

merely to help your memory :)

$$\vec{\nabla}\times\vec{V}(\vec{r}) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

a "formal expression"

$$\vec{\nabla}\times\vec{V} = \hat{e}_x\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_y & V_z \end{vmatrix} - \hat{e}_y\begin{vmatrix} \frac{\partial}{\partial x} & \partial/\partial z \\ V_x & V_z \end{vmatrix} + \hat{e}_z\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_x & V_y \end{vmatrix}$$

$$\vec{\nabla}\times\vec{V} = \hat{e}_x\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \ldots$$

no V_z in this example

$$\vec{V}(\vec{r}) = -y\hat{e}_x + x\hat{e}_y$$

$$\vec{\nabla} \times \vec{V}(\vec{r}) = \hat{e}_x\left(0 - \cancel{\frac{\partial x}{\partial z}}^{0}\right) - \hat{e}_y\left(0 + \cancel{\frac{\partial y}{\partial z}}^{0}\right) + \hat{e}_z\left(\overset{1}{\frac{\partial x}{\partial x}} - \overset{+1}{\frac{-\partial y}{\partial y}}\right)$$

$$\vec{\nabla} \times \vec{V}(\vec{r}) = 2\hat{e}_z$$

use right-hand rule to see that the direction really is in the z-direction