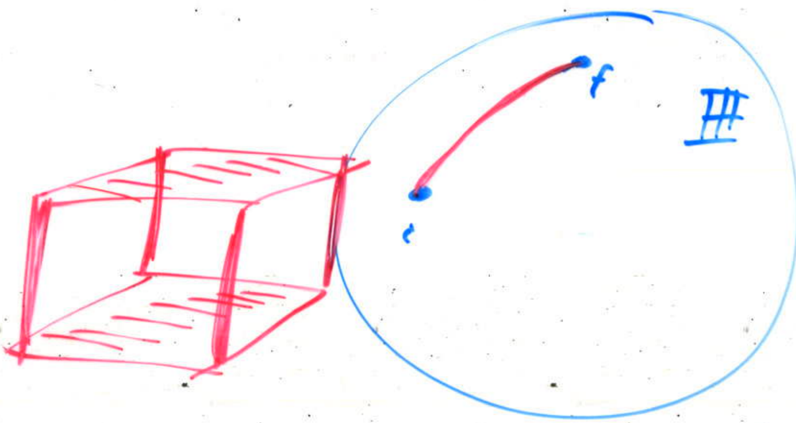
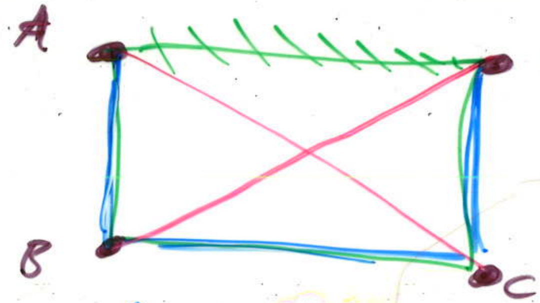
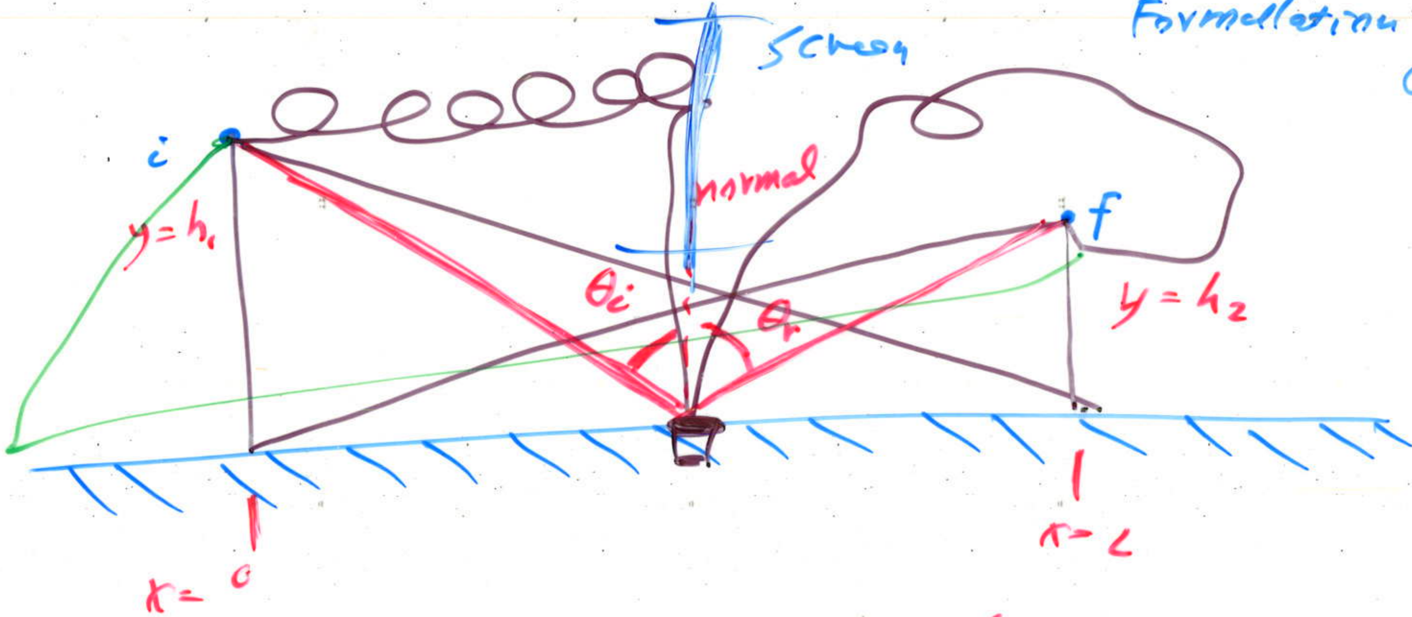
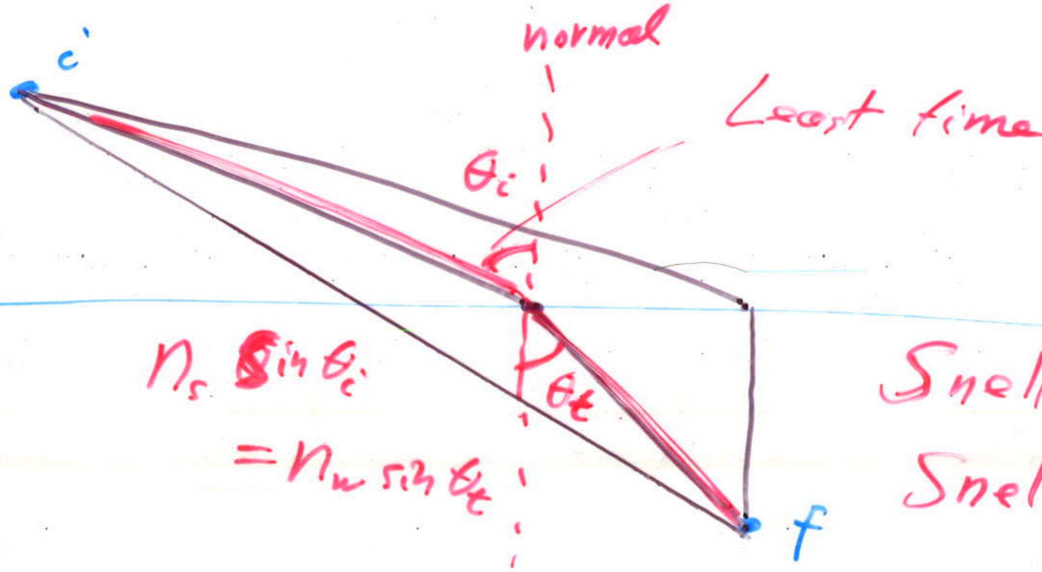
Calculus of Variations





Richard Feynman
Path Integral
Formulation of
QM





$V_s = \frac{c}{n_s}$

sand

water

$v_w < v_s$

~~$v_w = \frac{v_s}{n}$~~

$v_w = \frac{c}{n_w}$

$n_s \sin\theta_i = n_w \sin\theta_t$

Snell's law

Snel's law

scalar
function : number x $\longrightarrow \boxed{\text{function}} \longrightarrow$ number' $f(x)$

eg $2 \longrightarrow \boxed{x^2+1} \longrightarrow 5$

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functional I

function $f(x)$ $\longrightarrow \boxed{\text{functional}} \longrightarrow$ number I

$f(x)$	I
x^2+1	12
x^3+5	$9\frac{1}{2}$

$$I = \int_{x=x_1}^{x_2} G\left[f(x), \frac{df(x)}{dx}, \frac{d^2f(x)}{dx}, \ldots ; x\right] dx$$

explicit dependence on the integration variable x.

possibly more $\frac{d^3f(x)}{dx^3}, \ldots$

x is the variable
f is the function
G is the function of the function f and its derivatives

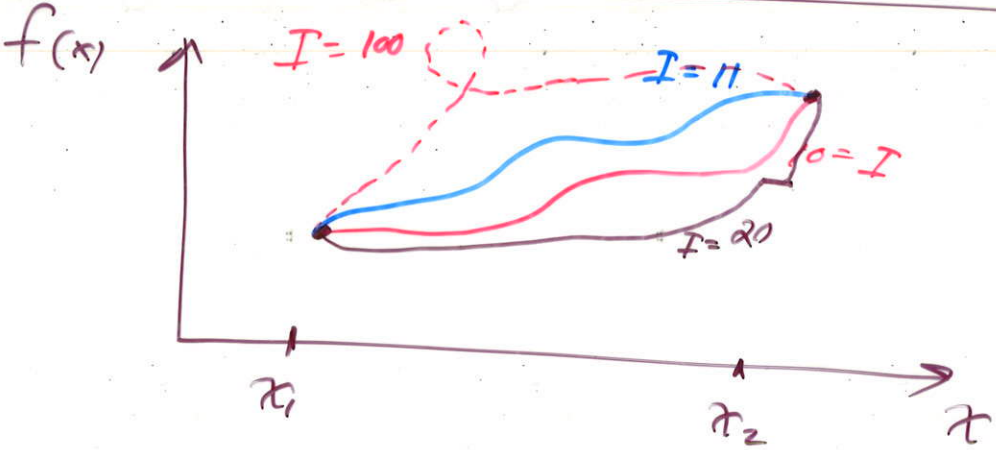
e.g. $f(x) = x^2+1$ $\quad f'(x) = 2x$ $\quad G = \frac{f^3}{f'} + \sin(f)\, x$

explicit x dependence

$$G = \frac{(x^2+1)^3}{2x} + \sin(x^2+1)\, x$$

$$I = \int_{x=x_1 \leftarrow 1}^{x_2 \leftarrow 2} \left[\frac{(x^2+1)^3}{2x} + \sin(x^2+1)\, x\right] dx = 13.1217\ldots$$

We look for function $f(x)$ that make the functional I stationary: first-order changes in $f(x)$ do not result in a first-order change in I.



call the function that makes I stationary (minimum here) $f_0(x)$.

$$f_\alpha(x) = f_0(x) + \alpha\, \eta(x) \qquad \eta(x_1) = 0 = \eta(x_2)$$

$$f'_\alpha(x) = \frac{d f_\alpha(x)}{dx} = f'_0(x) + \alpha\, \eta'(x)$$

We want $\delta I = 0 = \dfrac{\partial I}{\partial \alpha}\, d\alpha$

$$I = \int_{x=x_1}^{x_2} G\left[f_\alpha(x), f'_\alpha(x); x\right] dx$$

$$\delta I = \frac{\partial I}{\partial \alpha}\, d\alpha = \int_{x=x_1}^{x_2} \left[\left(\frac{\partial G}{\partial f_\alpha}\right) \underbrace{\left(\frac{\partial f_\alpha(x)}{\partial \alpha}\right)}_{\eta(x)} d\alpha + \left(\frac{\partial G}{\partial f'_\alpha}\right) \underbrace{\left(\frac{\partial f'_\alpha(x)}{\partial \alpha}\right)}_{\eta'(x) = \frac{d\eta}{dx}} d\alpha \right] dx$$

$$\delta I = \int_{x=x_1}^{x_2} \left[\frac{\partial G}{\partial f_\alpha}\, \eta(x) + \frac{\partial G}{\partial f'_\alpha}\, \eta'(x) \right] d\alpha\, dx$$

use integration by parts to move the derivative off $\eta(x)$, then both terms will be proportional to η.

$\cancel{\delta I = \int}$

2nd term

$$\int_{x=x_1}^{x_2} \frac{\partial G}{\partial f'_\alpha}\, \eta'(x)\, d\alpha\, dx = \cancel{\frac{\partial G}{\partial f'}\, \eta(x) \Big|_{x=x_1}^{x_2}}^{\,0} - \int_{x=x_1}^{x_2} \left[\frac{d}{dx}\left(\frac{\partial G}{\partial f'_\alpha}\right) \right] \eta(x)\, d\alpha\, dx$$

$$\boxed{\int_1^2 u\, dv = uv\Big|_1^2 - \int_1^2 v\, du}$$

"surface" term = zero

$$\delta I = \int_{x=x_1}^{x_2} \left[\frac{\partial G}{\partial f_\alpha} - \frac{d}{dx}\left(\frac{\partial G}{\partial f'_\alpha}\right) \right] \eta(x)\, d\alpha\, dx \overset{!}{=} 0 \quad \forall \eta(x)$$

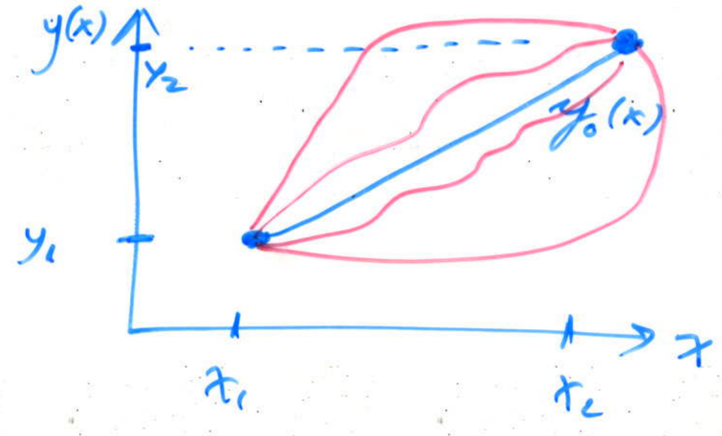
$$\Rightarrow \frac{\partial G}{\partial f} - \frac{d}{dx}\left(\frac{\partial G}{\partial f'}\right) = 0$$

Generates a differential equation that you will solve for $f_0(x)$.

Euler-Lagrange equation.

Examples

(1) Shortest distance between two points in the x-y plane



The length of the curve l (a number) will be the functional.

$$l = \int ds = \int \sqrt{dx^2 + dy^2} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$l = \int_{x=x_1}^{x_2} \sqrt{1 + y'^2}\; dx$$

$\leftarrow G[y(x), y'(x); x]$

missing   missing

$$\delta l = 0 \implies \text{Euler: } \frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0$$

$$\frac{\partial G}{\partial y} = 0 \quad \text{no } y\text{-dependence}$$

$$\frac{\partial G}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

$$\text{Euler: } 0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2}}\right] = 0$$

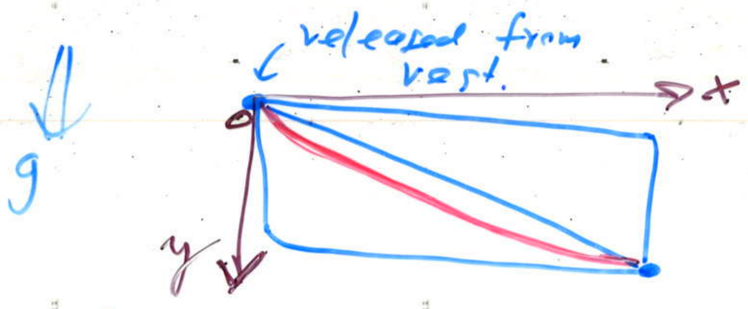
$$\implies \frac{y'}{\sqrt{1+y'^2}} = c = \text{constant} \qquad \text{solve for } y'$$

$$y' = \frac{c}{\sqrt{1-c^2}} = a \quad = \text{another constant}$$

$$\frac{dy(x)}{dx} = a \quad \Longrightarrow \quad y(x) = \overset{\text{slope}}{a}x + \overset{\text{y-intercept}}{b}$$

choose $\underline{a}$ and $\underline{b}$ to go through x_1 and x_2.

Brachistochrone (shortest time)



In a constant gravitational field, find the path $y(x)$ that minimizes the travel time from (x_1, y_1) to (x_2, y_2).

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

functional (I): t time $\quad G = ?$

$$t = \int dt = \int \frac{ds}{v}$$

Get the speed from energy conservation.

$$\frac{1}{2}\cancel{m} v^2 = \cancel{m} g y + \cancel{\text{const}}^{\,0}$$

$$v = \sqrt{2gy}$$

$$t = \int_{x=x_1}^{x_2} \frac{\sqrt{1+y'^2}\, dx}{\sqrt{2gy}}$$

$$G[y(x), y'(x); x] = \sqrt{\frac{1+y'^2}{2gy}}$$

$$\delta t = 0 \Longrightarrow \frac{\partial G}{\partial y} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0$$

$$\left(\frac{\partial G}{\partial y}\right)_{y'} = \frac{-1}{2}\sqrt{\frac{1+y'^2}{2gy^3}}$$

$$\left(\frac{\partial G}{\partial y'}\right)_y = \frac{1}{\sqrt{2gy}}\,\cancel{\tfrac{1}{2}}\,(1+y'^2)^{-1/2}\,\cancel{2}y' = \frac{1}{\sqrt{2gy_{(x)}}}\frac{y'_{(x)}}{\sqrt{1+y'^2_{(x)}}}$$

Euler-Lagrange: $\dfrac{\partial G}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial G}{\partial y'}\right) = 0$

$$-\frac{1}{2}\sqrt{\frac{1+y'^2_{(x)}}{2gy^3_{(x)}}} - \frac{d}{dx}\left[\frac{1}{\sqrt{2gy_{(x)}}}\frac{y'_{(x)}}{\sqrt{1+y'^2_{(x)}}}\right] = 0$$

2nd order. <u>linear?</u>

Two ways to possibly generate an easier D.E.

① Beltrami form of the Euler equation.

First form: $$\frac{\partial G}{\partial y} - \frac{d}{dx}\left[\frac{\partial G}{\partial y'}\right] = 0$$

Consider

$$\frac{dG}{dx} = \frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\underbrace{\frac{dy}{dx}}_{y'} + \frac{\partial G}{\partial y'}\underbrace{\frac{dy'}{dx}}_{y'' = \frac{d^2y}{dx^2}}$$

$$\frac{dG}{dx} = \frac{\partial G}{\partial x} + \underbrace{\frac{\partial G}{\partial y}y'}_{\text{solve for this}} + \frac{\partial G}{\partial y'}y''$$

$$\frac{\partial G}{\partial y}y' = \frac{dG}{dx} - \frac{\partial G}{\partial x} - \frac{\partial G}{\partial y'}y''$$

$$\frac{\partial G}{\partial y}y' = \left(\frac{d}{dx}\left[\frac{\partial G}{\partial y'}\right]\right)y' \leftarrow \text{from first form}$$

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subtract

$$0 = \underbrace{\frac{dG}{dx} - \frac{\partial G}{\partial y'}y'' - \left(\frac{d}{dx}\left[\frac{\partial G}{\partial y'}\right]\right)y'}\; - \frac{\partial G}{\partial x}$$

$$\boxed{0 = \frac{d}{dx}\left[G - \frac{\partial G}{\partial y'}y'\right] - \frac{\partial G}{\partial x}}$$

useful if G does not depend on x explicitly.

Beltrami form of Euler equation

Special case: $\frac{\partial G}{\partial x} = 0$

$$0 = \frac{d}{dx}\left[G - \frac{\partial G}{\partial y'} y'\right] \Longrightarrow G - \frac{\partial G}{\partial y'} y' = \text{constant}$$

$$G = \sqrt{\frac{1 + y'^2}{2gy}}$$

$$G - \frac{\partial G}{\partial y'} y' = \text{constant} = C \Longrightarrow \sqrt{\frac{1+y'^2}{2gy}} - \frac{y'^2}{\sqrt{2gy}\sqrt{1+y'^2}} = C$$

$$\Rightarrow \frac{(1+y'^2) - y'^2}{\sqrt{2gy}\sqrt{1+y'^2}} = C$$

$$\Longrightarrow \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = C$$ square both sides

$$\Longrightarrow \frac{1}{2gy} \cdot \frac{1}{(1+y'^2)} = d$$ invert both sides

$$y(1+y'^2) = \frac{1}{d\,2g} = a$$

$$y\left(1 + \left[\frac{dy}{dx}\right]^2\right) = a \Longrightarrow \frac{dy}{dx} = \sqrt{\frac{a}{y} - 1}$$

parametric solution

$$\left.\begin{aligned} x(\theta) &= \frac{a}{2}(\theta - \sin\theta) + b \\ y(\theta) &= \frac{a}{2}(1 - \cos\theta) \end{aligned}\right\}$$ cycloid

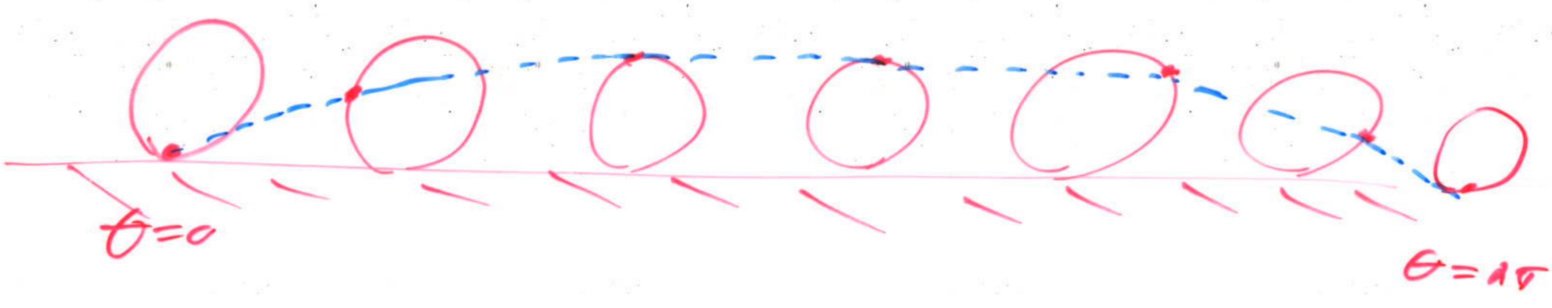
check

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cancel{\frac{a}{2}}\sin\theta}{\cancel{\frac{a}{2}}(1-\cos\theta)} \stackrel{?}{=} \sqrt{\frac{2}{(1-\cos\theta)} - 1}$$

RHS :

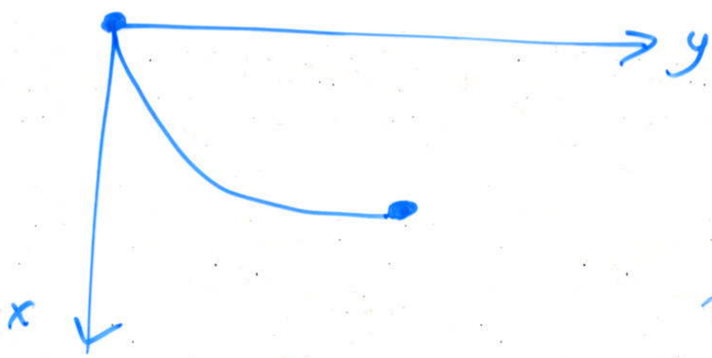
$$\sqrt{\frac{\overset{1}{\cancel{2-1}} + \cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}\cdot\left(\frac{1-\cos\theta}{1-\cos\theta}\right)}$$

$$= \frac{\sqrt{1-\cos^2\theta}}{\sqrt{(1-\cos\theta)^2}} = \frac{\sin\theta}{1-\cos\theta} \quad \checkmark$$



$$\left.\frac{dy}{dx}\right|_{\theta=0} = \infty$$

② Transpose x and y



$$t = \int dt = \int \frac{ds}{v}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2}\,dx$$

$$v = \sqrt{2gx} \qquad = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$t = \int_{x_1}^{x_2} \sqrt{\frac{1 + y'^2}{2gx}}\,dx \quad \Bigg\| \quad G = \sqrt{\frac{1+y'^2}{2gx}}$$

First Form of Euler: $\cancelto{0}{\frac{\partial G}{\partial y}} - \frac{d}{dx}\left(\frac{\partial G}{\partial y'}\right) = 0$

$$C = \frac{\partial G}{\partial y'} = \frac{1}{\sqrt{2gx}}\,\frac{1}{\cancel{2}}\left(1 + y'^2\right)^{-1/2}\cancel{2}y' = \frac{y'}{\sqrt{2gx}\sqrt{1+y'^2}} = C$$

$$\frac{y'}{\sqrt{1+y'^2}} = c\sqrt{2gx}$$

$$\frac{dy}{dx} = y' = \frac{x}{\sqrt{ax - x^2}} \longrightarrow \int dy = \int \frac{x\,dx}{\sqrt{ax - x^2}}$$

substitution: $x = \frac{a}{2}(1 - \cos\theta)$

$$y = \int \frac{a}{2}(1 - \cos\theta)\,d\theta = \frac{a}{2}\theta - \frac{a}{2}\sin\theta + \overset{b}{\text{constant}}$$