Curl $\vec{\nabla}\times$

Acts on a vector field and produces another vector field.

$$\vec{\nabla}\times\vec{V}(\vec{r}) = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{e}_1h_1 & \hat{e}_2h_2 & \hat{e}_3h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ V_1h_1 & V_2h_2 & V_3h_3 \end{vmatrix}$$

* where V_i depends on $(q_1, q_2 \& q_3)$

* and scale factors depend on $(q_1, q_2 \& q_3)$ too

$[\vec{\nabla}\times\vec{V}(\vec{r})]_1$

component of $\vec{\nabla}\times\vec{V}$ along the $\hat{e}_1$ direction

$$[\vec{\nabla}\times\vec{V}(\vec{r})]_1 = \frac{1}{h_1h_2h_3}\hat{e}_1h_1\begin{vmatrix} \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_2V_2 & h_3V_3 \end{vmatrix}$$

$$= \frac{1}{h_1h_2h_3}\hat{e}_1h_1\left[\frac{\partial h_3V_3}{\partial q_2} - \frac{\partial h_2V_2}{\partial q_3}\right]$$

* & don't forget h_i & V_i can be dependent on all q_i's!

Laplacian ∇^2 __

* must act on something (specifically a scalar field, to produce... another scalar field)

eg. $\Phi(\vec{r})$

$\vec{\nabla}\cdot(\vec{\nabla}\Phi(\vec{r})) = \text{div}[\text{grad}(\Phi(\vec{r}))] = \underline{\underline{\nabla^2\Phi(\vec{r})}}$

* can only be done in this order

Cartesian: $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

& remember, h$_i$'s aren't necessarily constant!

General:

$$\nabla^2 \Phi(q_1,q_2,q_3) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial q_1}\Phi\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial q_2}(\Phi)\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial q_3}\Phi\right)\right]$$

* no real physical description of Laplacian ...
→ used to find sources & sinks in fields

& then there are 2 trivial combo's that always equal zero:

$$\vec{\nabla}\times\left(\vec{\nabla}\Phi(\vec{r})\right) = \text{curl}(\text{grad}(\Phi(\vec{r}))) = \vec{0}$$

for All functions $\forall \Phi, \forall(\vec{r})$
– gradient has no curl –

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{V}(\vec{r})\right) = 0 \quad \forall \vec{V}, \forall(\vec{r})$$

(vector)

curl has no divergence

---

eg $\frac{\partial}{\partial x}\frac{\partial}{\partial y}(?) - \frac{\partial}{\partial y}\frac{\partial}{\partial x}(?)$

Separation of Variables

Linear Partial differential equations

Examples:

Heat Equation } dependent on 3 space variables & time
Schrödinger Equation }
Wave Equation }electromagnetic (not water waves b/c its viscous)
Laplace's Equation:

$$\nabla^2 \Phi(\vec{r}) = 0$$

a linear, second order, homogeneous, partial diffeq

Poisson's Eqn: $\nabla^2 \Phi(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$ (charge density) ← coordinate-free

cartesian $r = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$
spherical $r = r\hat{e}_r + \theta\hat{e}_\theta + \phi\hat{e}_\phi$

In general coordinates:

$$\nabla^2 \Phi(\vec{r}) = \frac{1}{h_1 h_2 h_3} \sum_{i=1}^{3} \frac{\partial}{\partial q_i}\left[\frac{h_1 h_2 h_3}{h_i^2} \frac{\partial}{\partial q_i} \Phi(q_1, q_2, q_3)\right] = 0$$

↓ comes from divergence

Cartisian Coordinates (all h's are 1)

$$\nabla^2 \Phi(\vec{r}) = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \Phi(x, y, z)$$

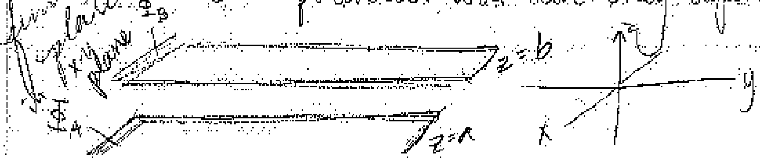
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \Phi(x, y, z) = 0$$

example: Boundary Conditions depend on only one variable z

potential will also only depend on z

b/c of the symmetry, no dependence on x, y

infinite plates (xy plane)



Solve for $\Phi(\vec{r})$ between the plates
Φ can depend only on z

$\Phi(z)$
now apply Laplace's Eqn to potential

$$\nabla^2 \Phi(\vec{r}) = \nabla^2 \Phi(z) = \frac{\partial^2}{\partial z^2}\Phi(z) = 0$$

$$= \frac{d^2}{dz^2}\Phi(z) = 0 \quad \therefore \underline{\Phi(z) = cz + d}$$

b/c Φ is fully dependent on z, so it's an ordinary

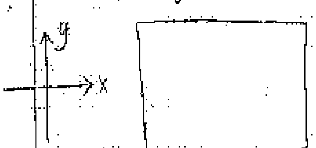
$\Phi(z) = cz + d$ & fix c and d with boundary conditions

$$\boxed{\begin{aligned} \Phi(a) &= \Phi_a = ca + d \\ \Phi(b) &= \Phi_b = cb + d \end{aligned}}$$

2 equations & 2 unknowns

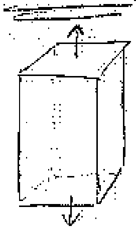
Now let Boundary Conditions depend on 2 coordinates (x and y)

top view



cross-sectional area

side view



infinite in z direction

→ can't tell if you've moved up/down

* Goal: find $\Phi(\vec{r})$ inside the channel

$\Phi(x,y)$

$$\nabla^2 \Phi(\vec r) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Phi(x,y) = 0$$

Assume the solution is $\Phi(x,y) = f(x)\,g(y)$
or a guess rather ☺

$$\nabla^2 \Phi(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x)\,g(y) = 0$$

$$\frac{d^2 f(x)}{dx^2}\,g(y) + f(x)\,\frac{d^2 g(y)}{dy^2}$$

<u>trick</u>: divide by $f(x)g(y)$

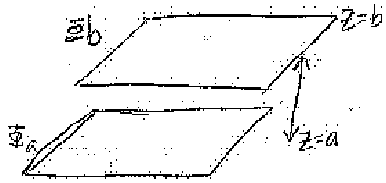
$$\frac{\frac{d^2 f(x)}{dx^2}\,g(y)}{f(x)\,g(y)} + \frac{f(x)\,\frac{d^2 g(y)}{dy^2}}{f(x)\,g(y)} = \frac{0}{f(x)\cdot g(y)} = 0$$

$$\frac{\frac{d^2 f(x)}{dx^2}}{f(x)} + \frac{\frac{d^2 g(y)}{dy^2}}{g(y)} = 0$$

$$\frac{\frac{d^2 f(x)}{dx^2}}{f(x)} = \boxed{-\frac{\ddot g(y)}{g(y)} = \frac{\ddot f(x)}{f(x)}}$$

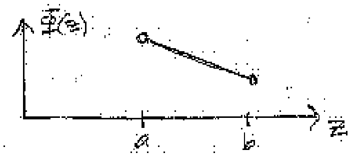
& they're separated

Boundary Conditions Depend on one Variable



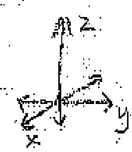
$$\nabla^2 \Phi(\vec{r}) = \frac{d^2\Phi(z)}{dz^2} = 0$$

$\Phi(z) = Cz + d$ (as seen earlier)



Boundary Conditions (B.C.) depend on 2 variables (x, y)

Voltages can't be dependent on z, so they look like stripes of voltage



$$\nabla^2 \Phi(\vec{r}) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \Phi(x,y) = 0$$

$$\Phi(x,y) = f(x)\,g(y)$$

$$\frac{\frac{d^2 f(x)}{dx^2}}{f(x)} = -\frac{\frac{d^2 g(y)}{dy^2}}{g(y)}$$

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)}$$ equality must hold true for any $f(x)$ and $g(y)$ you pick

$\Downarrow$

for this to hold, they must both be constants!

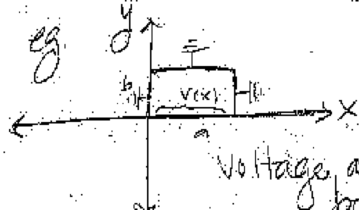
$$\frac{f''(x)}{f(x)} = \text{separation constant} = \frac{-g''(y)}{g(y)}$$

in math, this can be anything, but in physics, it can't equal zero b/c then

$$\frac{f''(x)}{f(x)} = 0, \text{ letting } f''(x) = 0 \ \& \ f(x) = ax + b \ \& \ g(y) = cy + d$$

however, these don't match the boundary conditions

eg.



voltage along the boundaries are arbitrary, so we choose 3 of the sides to be grounded (V=0) and the 4th to be V(x)

B.C.'s:

$\Phi(x, b) = 0$ (top)
$\Phi(0, y) = 0$ (left)
$\Phi(a, y) = 0$ (right)
$\Phi(x, 0) = V(x)$ (bottom)
↳ consider this to be tune-able

goal: solve for $\Phi(x,y)$ inside

let's let $V(x) = 20V$

If you make the separation constant positive,

$$\frac{f''(x)}{f(x)} = \alpha^2 = -\frac{g''(y)}{g(y)}$$

$f''(x) = \alpha^2 f(x)$, $f''(x) - \alpha^2 f(x)$ ⇒ exponentials

add them back together w/ superpositions
orthogonals

$f''(x) - \alpha^2 f(x) = 0$

$f(x) = \underline{Ae^{\alpha x} + Be^{-\alpha x}}$

There's no way for both of these to vanish together so they can't meet the boundary conditions

$-\dfrac{g''(y)}{g(y)} = \alpha^2$

$g''(y) + \alpha^2 g(y) = 0$

$g(y) = C\sin(\alpha y) + D\cos(\alpha y)$

Arguments against a positive separation constant:

① $\Phi(x,y)$ must vanish @ $x=0$ & $x=a$ to satisfy boundary conditions & exponentials can't do that

② $V(x)$ must be made of the $f(x)$ functions and $e^{\alpha x}, e^{-\alpha x}$ are not complete.

Therefore, Separation constant is negative

$\Longrightarrow f(x) = A\sin(\alpha x) + B\cos(\alpha x)$
$g(y) = Ce^{\alpha y} + De^{-\alpha y}$

Use $\overset{\text{boundary conditions}}{\text{B.C.}}$ to eliminate some A, B, C, D.

left side $\Phi(0,y) = 0$

$\Phi(x,y) = f(x)g(y) = [A\sin(\alpha x) + B\cos(\alpha x)] \cdot [Ce^{\alpha y} + De^{-\alpha y}]$

$\Phi(0,y) = B \cdot \underbrace{[Ce^{\alpha y} + De^{-\alpha y}]}_{} = 0$

b/c $x=0$ makes $A\sin(\alpha x)$ go away

can't equal zero b/c then the boundary conditions wouldn't be satisfied

$\therefore \underline{B=0}$

$x = a$ right side

$\Phi(a,y) = 0$

$\Phi(x,y) = A\sin(\alpha x)\left[Ce^{\alpha y} + De^{-\alpha y}\right]$

$\Phi(a,y) = A\sin(\alpha a)\left[Ce^{\alpha y} + De^{-\alpha y}\right] = 0$

& from the previous argument,

$A\sin(\alpha a) = 0$

A can't equal zero b/c it wouldn't satisfy B.C.

$\sin(\alpha a) = 0$

$\alpha a = n\pi, \quad n = 1, 2, 3, \ldots$

$n \neq 0$

$\boxed{\therefore \alpha = \frac{n\pi}{a}}$ Quantization Condition

alpha is now discrete, rather than continuous

(B still equals zero)

$\Phi(x,y) = A\sin\left(\frac{n\pi x}{a}\right)\left[Ce^{\frac{n\pi}{a}y} + De^{-\frac{n\pi y}{a}}\right]$

at $y = b$

$\Phi(x,b) = A\sin\left(\frac{n\pi x}{a}\right)\left[Ce^{\frac{n\pi b}{a}} + De^{-\frac{n\pi b}{a}}\right] = 0$

$Ce^{\frac{n\pi b}{a}} = -De^{-\frac{n\pi b}{a}}$

$C = -De^{-\frac{2\pi b n}{a}}$

---

factor out D!

$\Phi(x,y) = \overset{F}{AD}\sin\left(\frac{n\pi x}{a}\right)\left[\overset{(C)}{\left(-\cancel{D}e^{-\frac{2\pi b n}{a}}\right)}e^{\frac{n\pi y}{a}} + \cancel{D}e^{-\frac{n\pi y}{a}}\right]$

$\Phi(x,y) = F_n\sin\left(\frac{n\pi x}{a}\right)\left[-e^{-\frac{2\pi b n}{a}}e^{\frac{n\pi y}{a}} + e^{-\frac{n\pi y}{a}}\right]$

now we need to specify F

where $n = 1, 2, 3, \ldots$

General Solution to $\nabla^2\Phi(\vec{r}) = 0$

$$\Phi(x,y) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{a}\right)\left[-e^{-\frac{2\pi b n}{a}} e^{\frac{n\pi y}{a}} + e^{-\frac{n\pi y}{a}}\right]$$

One more boundary condition

$\Phi(x,0) = V(x) \leftarrow$ arbitrary

$$V(x) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{a}\right)\left[-e^{-\frac{2\pi b n}{a}}(1) + (1)\right]$$

*almost a fourier decomposition

---

recall:

$$\hat{e}_k \cdot \vec{V} = \sum_{n=1} F_n \underbrace{\hat{e}_n \cdot \hat{e}_k}_{\delta_{nk}} \qquad F_n = \hat{e}_n \cdot \vec{V}$$

now all n's are changed to k's

---

need function analog of the above

sin & cos are the basis functions

$T = 2a$

$$V(x) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{a}\right)\left[1 - e^{-\frac{2\pi b n}{a}}\right]$$

multiply both sides by sin of a diff. frequency

$$\int_0^{2a} \sin\left(\frac{k\pi x}{a}\right) V(x)\,dx = \sum_{n=1}^{\infty} \int_0^{2a} F_n \sin\left(\frac{n\pi x}{a}\right)\left[1 - e^{-\frac{2\pi b n}{a}}\right] \sin\left(\frac{k\pi x}{a}\right)dx$$

$$\omega = \frac{\pi}{a} \qquad T = \frac{2\pi}{\omega} = 2a$$

let n=1 for determining T

$$= \sum_{n=1}^{\infty} F_n \left[1 - e^{-\frac{2\pi b n}{a}}\right] \underbrace{\int_{x=0}^{2a} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{k\pi x}{a}\right)dx}_{C_{nk}}$$

← you need a normalization constant out front (a) kronecker delta

$$\sum_{n=1}^{\infty} F_n\left[1-e^{-\frac{2\pi b n}{a}}\right] a\,\delta_{nk} = F_k\left[1-e^{-\frac{2\pi b k}{a}}\right] a$$

↑ change all n's to k's & get rid of sum

Bessel functions have n dependence, but cos & sin do not! ☺

$$F_k = \frac{\int_0^{2a} \sin\left(\frac{k\pi x}{a}\right) V(x)\,dx}{a\left[1-e^{-\frac{2\pi b k}{a}}\right]}$$

now that you know F_n, you can determine $\Phi(x,y)$

If one of the right or left sides had been of some $V(x)$, you would've switched the x & y on the function above (vice & back some...)

Also, if more than 1 side had an arbitrary $V(x)$, you would solve for each individually & then add them up (superposition)

~ Bring questions about homework for Wed ~