Curvilinear Coordinates   2-dimensional

Polar Coordinates

$r = \sqrt{x^2+y^2} \geq 0$

$\varphi = \arctan\left(\frac{y}{x}\right)$

maybe $\pm \pi$ rad

Unit vectors

$\hat{e}_r = \hat{r}$ = points in direction of increasing r

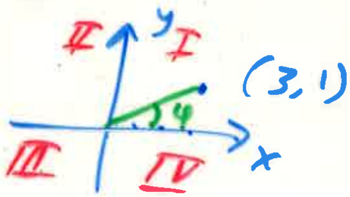
$\hat{e}_\varphi = \hat{\varphi}$ = " " φ

orthonormal

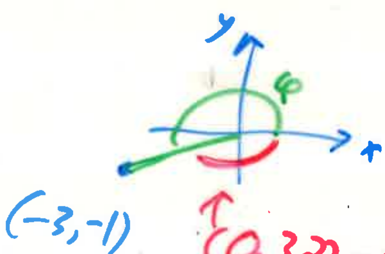
$\hat{r}\cdot\hat{r} = 1 = \hat{\varphi}\cdot\hat{\varphi}$

$\hat{r}\cdot\hat{\varphi} = 0$

e.g.



$\varphi = \arctan\left(\frac{1}{3}\right) = .322 \text{ rad} = 18.4°$



$(0.322-\pi)$ rad

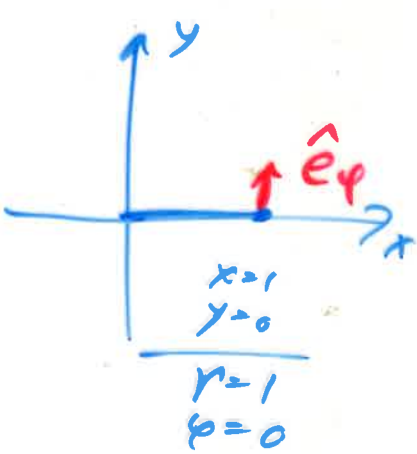
$\varphi = \arctan\left(\frac{-1}{-3}\right) = .322 \text{ rad} + \pi \text{ rad} = 18.4° + 180°$

$x = r\cos\varphi$

$y = r\sin\varphi$

$\hat{e}_x = \hat{e}_1 = \hat{x} = \hat{\imath}$

$\hat{e}_y = \hat{e}_2 = \hat{y} = \hat{\jmath}$



$\hat{e}_r = \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{e}_x + y\hat{e}_y}{\sqrt{x^2+y^2}}$

$\hat{e}_\varphi = \hat{\varphi} = \frac{-y\hat{e}_x + x\hat{e}_y}{\sqrt{x^2+y^2}}$

Functions of time $x(t)$, $y(t)$, $r(t)$, $\varphi(t)$.

$$\dot{r} = \frac{dr}{dt} = \frac{d}{dt}(x^2+y^2)^{1/2} = \frac{1}{2}(x^2+y^2)^{-1/2}(2x\dot{x}+2y\dot{y})$$

$$= \frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}} \neq |\dot{\vec{r}}| = |\vec{v}| = \sqrt{\dot{x}^2+\dot{y}^2}$$

$$\boxed{\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y} \qquad [\dot{r}] = \frac{L}{T}$$

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{d}{dt}\left[\arctan\left(\frac{y}{x}\right)\right] = \frac{\frac{d}{dt}\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2}$$

$$= \frac{1}{1+\left(\frac{y^2}{x^2}\right)} \cdot \frac{x\dot{y}-y\dot{x}}{x^2} = \frac{x\dot{y}-y\dot{x}}{x^2+y^2}$$

$$[\dot{\varphi}] = \frac{1}{T}$$

$$\dot{\hat{e}}_r = \frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\frac{x\hat{e}_x+y\hat{e}_y}{\sqrt{x^2+y^2}}\right) \qquad \dot{\hat{e}}_x = 0 = \dot{\hat{e}}_y$$

$$= \frac{d}{dt}\left[(x\hat{e}_x+y\hat{e}_y)\cdot(x^2+y^2)^{-1/2}\right]$$

$$= (\dot{x}\hat{e}_x+\dot{y}\hat{e}_y)(x^2+y^2)^{-1/2} + (x\hat{e}_x+y\hat{e}_y)\left(-\cancel{\tfrac{1}{2}}\right)(x^2+y^2)^{-3/2}\cancel{2}(x\dot{x}+y\dot{y})$$

$$\dot{\hat{e}}_r = \frac{(\dot{x}\hat{e}_x + \dot{y}\hat{e}_y)(x^2+y^2) - (x\hat{e}_x + y\hat{e}_y)(x\dot{x} + y\dot{y})}{(x^2+y^2)^{3/2}}$$

$$= \frac{\cancel{\dot{x}x^2\hat{e}_x} + \dot{y}x^2\hat{e}_y + \dot{x}y^2\hat{e}_x + \cancel{\dot{y}y^2\hat{e}_y} - \cancel{\dot{x}x^2\hat{e}_x} - \cancel{\dot{y}y^2\hat{e}_y} - xy\dot{y}\hat{e}_x - xy\dot{x}\hat{e}_y}{(x^2+y^2)^{3/2}}$$

$$= \frac{\dot{y}x^2\hat{e}_y + \dot{x}y^2\hat{e}_x - xy\dot{y}\hat{e}_x - xy\dot{x}\hat{e}_y}{(x^2+y^2)^{3/2}}$$

$$= \frac{(-y\hat{e}_x + x\hat{e}_y)}{\sqrt{x^2+y^2}} \cdot \frac{(x\dot{y} - y\dot{x})}{x^2+y^2} = \boxed{\hat{e}_\varphi \cdot \dot{\varphi} = \dot{\hat{e}}_r}$$

HW $\dot{\hat{e}}_\varphi = ?$

Cartesian Coord's

Position vector: $\vec{r} = x\hat{e}_x + y\hat{e}_y$

Velocity vector: $\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} = \dot{x}\hat{e}_x + \dot{y}\hat{e}_y$

Acceleration vector: $\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{x}\hat{e}_x + \ddot{y}\hat{e}_y$

Polar Coord's

Position vector: $\vec{r} = r\hat{e}_r + \cancel{\varphi\hat{e}_\varphi}$

Velocity vector: $\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \frac{d}{dt}[r\hat{e}_r]$

$= \dot{r}\hat{e}_r + r\dot{\hat{e}}_r = \underbrace{\dot{r}}_{\text{radial speed}}\hat{e}_r + \underbrace{r\dot{\varphi}}_{\text{tangential speed}}\hat{e}_\varphi$

E.g. In astronomy: radial speed → Doppler Effect; tangential speed → proper Motion.

HW: Acceleration vector: $\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}}$

$= \underbrace{\underline{\qquad\qquad}}_{\text{radial accel.}}\hat{e}_r + \underbrace{\underline{\qquad\qquad}}_{\text{tangential accel.}}\hat{e}_\varphi$

¿ + (Centripetal)

(~~centripetal~~)