Chapter 12

Fourier Series

Just before 1800, the French mathematician/physicist/engineer Jean Baptiste Joseph Fourier made an astonishing discovery. As a result of his investigations into the partial differential equations modeling vibration and heat propagation in bodies, Fourier was led to claim that "every" function could be represented by an infinite series of elementary trigonometric functions — sines and cosines. As an example, consider the sound produced by a musical instrument, e.g., piano, violin, trumpet, oboe, or drum. Decomposing the signal into its trigonometric constituents reveals the fundamental frequencies (tones, overtones, etc.) that are combined to produce its distinctive timbre. The Fourier decomposition lies at the heart of modern electronic music; a synthesizer combines pure sine and cosine tones to reproduce the diverse sounds of instruments, both natural and artificial, according to Fourier's general prescription.

Fourier's claim was so remarkable and unexpected that most of the leading mathematicians of the time did not believe him. Nevertheless, it was not long before scientists came to appreciate the power and far-ranging applicability of Fourier's method, thereby opening up vast new realms of physics, engineering, and elsewhere, to mathematical analysis. Indeed, Fourier's discovery easily ranks in the "top ten" mathematical advances of all time, a list that would include Newton's invention of the calculus, and Gauss and Riemann's establishment of differential geometry that, 70 years later, became the foundation of Einstein's general relativity. Fourier analysis is an essential component of much of modern applied (and pure) mathematics. It forms an exceptionally powerful analytical tool for solving a broad range of partial differential equations. Applications in pure mathematics, physics and engineering are almost too numerous to catalogue — typing in "Fourier" in the subject index of a modern science library will dramatically demonstrate just how ubiquitous these methods are. Fourier analysis lies at the heart of signal processing, including audio, speech, images, videos, seismic data, radio transmissions, and so on. Many modern technological advances, including television, music CD's and DVD's, video movies, computer graphics, image processing, and fingerprint analysis and storage, are, in one way or another, founded upon the many ramifications of Fourier's discovery. In your career as a mathematician, scientist or engineer, you will find that Fourier theory, like calculus and linear algebra, is one of the most basic and essential tools in your mathematical arsenal. Mastery of the subject is unavoidable.

Furthermore, a remarkably large fraction of modern pure mathematics is the result of subsequent attempts to place Fourier series on a firm mathematical foundation. Thus, all of the student's "favorite" analytical tools, including the definition of a function, the ε–δ definition of limit and continuity, convergence properties in function space, including uniform convergence, weak convergence, etc., the modern theory of integration and measure, generalized functions such as the delta function, and many others, all owe a profound debt to the prolonged struggle to establish a rigorous framework for Fourier analysis. Even more remarkably, modern set theory, and, as a result, mathematical logic and foundations, can be traced directly back to Cantor's attempts to understand the sets upon which Fourier series converge!

As we will appreciate, Fourier series are, in fact, a very natural outgrowth of the basic linear algebra constructions that we have already developed for analyzing discrete dynamical processes. The Fourier representation of a function is a continuous counterpart of the eigenvector expansions used to solve linear systems of ordinary differential equations. The fundamental partial differential equations governing heat propagation and vibrations in continuous media can be viewed as the function space counterparts of such discrete systems. In the continuous realm, solutions are expressed as linear combinations of simple "separable" solutions constructed from the eigenvalues and eigenfunctions of an associated self-adjoint boundary value problem. The efficacy of Fourier analysis rests on the orthogonality properties of the trigonometric functions, which is a direct consequence of their status as eigenfunctions. So, Fourier series can be rightly viewed as a function space version of the finite-dimensional spectral theory of symmetric matrices and orthogonal eigenvector bases. The main complication is that we must now deal with infinite series rather than finite sums, and so convergence issues that do not appear in the finite-dimensional situation become of paramount importance.

Once we have established the proper theoretical background, the trigonometric Fourier series will no longer be a special, isolated phenomenon, but, rather, in its natural context as the *simplest* representative of a broad class of orthogonal eigenfunction expansions based on self-adjoint boundary value problems. Modern and classical extensions of the Fourier method, including Fourier integrals, discrete Fourier series, wavelets, Bessel functions, spherical harmonics, as well as the entire apparatus of modern quantum mechanics, all rest on the same basic theoretical foundation, and so gaining familiarity with the general theory and abstract eigenfunction framework will be essential. Many of the most important cases used in modern physical and engineering applications will appear in the ensuing chapters.

We begin our development of Fourier methods with a section that will explain why Fourier series naturally appear when we move from discrete systems of ordinary differential equations to the partial differential equations that govern the dynamics of continuous media. The reader uninterested in motivations can safely omit this section as the same material reappears in Chapter 14 when we completely analyze the dynamical partial differential equations that lead to Fourier methods. Beginning in Section 12.2, we shall review, omitting proofs, the most basic computational techniques in Fourier series, for both ordinary and generalized functions. In the final section, we include an abbreviated introduction to the analytical background required to develop a rigorous foundation for Fourier series methods.

12.1. Dynamical Equations of Continuous Media.

The purpose of this section is to discover why Fourier series arise naturally when we move from discrete systems of ordinary differential equations to the partial differential equations that govern the dynamics of continuous mechanical systems. In our reconstrucd-tion of Fourier's thought processes, let us start by reviewing what we have learned.

In Chapter 6, we characterized the equilibrium equations of discrete mechanical and electrical systems as a linear algebraic system

$$K\mathbf{u} = \mathbf{f} \tag{12.1}$$

with symmetric, positive (semi-)definite coefficient matrix K. There are two principal types of dynamical systems associated with such equilibrium equations. Free vibrations are governed by Newton's Law, which leads to a second order system of ordinary differential equations (9.59), of the form

$$\frac{d^2\mathbf{u}}{dt^2} = -K\mathbf{u}. \tag{12.2}$$

On the other hand, the gradient flow equations (9.21), namely

$$\frac{d\mathbf{u}}{dt} = -K\mathbf{u}, \tag{12.3}$$

are designed to decrease the quadratic energy function $q(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T K\mathbf{u}$ as rapidly as possible. In each case, the solution to the system was made by imposing a particular *ansatz* or inspired guess[†] for the basic solutions. In the case of a gradient flow, the solutions are of exponential form $e^{-\lambda t}\mathbf{v}$, while for vibrations they are of trigonometric form $\cos(\omega t)\,\mathbf{v}$ or $\sin(\omega t)\,\mathbf{v}$ with $\omega^2 = \lambda$. In either case, substituting the relevant solution ansatz reduces the dynamical system to the algebraic eigenvalue problem

$$K\mathbf{v} = \lambda\mathbf{v} \tag{12.4}$$

for the matrix K. Each eigenvalue and eigenvector creates a particular solution or natural mode, and the general solution to the dynamical system can be expressed as a linear superposition of these fundamental modes. The remarkable fact is that the *same* mathematical framework, suitably reinterpreted, carries over directly to the continuous realm!

In Chapter 11, we developed the equilibrium equations governing one-dimensional continuous media — bars, beams, etc. The solution is now a function $u(x)$ representing, say, displacement of the bar, while the positive (semi-)definite matrix is replaced by a certin positive (semi-)definite linear operator $K[u]$. Formally, then, under an external forcing function $f(x)$, the equilibrium system can be written the abstract form

$$K[u] = f. \tag{12.5}$$

By analogy with (12.1, 3), the corresponding gradient flow system will thus be of the form[‡]

$$\frac{\partial u}{\partial t} = -K[u]. \tag{12.6}$$

[†] See the footnote in Example 7.32 for an explanation of this term.

[‡] Since $u(t,x)$ now depends upon time as well as position, we switch from ordinary to partial derivative notation.

Such partial differential equations model diffusion processes in which a quadratic energy functional is decreasing as rapidly as possible. A good physical example is the flow of heat in a body; the heat disperses throughout the body so as to decrease the thermal energy as quickly as it can, tending (in the absence of external heat sources) to thermal equilibrium. Other physical processes modeled by (12.6) include diffusion of chemicals (solvents, pollutants, etc.), and of populations (animals, bacteria, people, etc.).

The simplest and most instructive example is a uniform periodic (or circular) bar of length 2π. As we saw in Chapter 11, the equilibrium equation for the temperature $u(x)$ takes the form

$$K[u] = -u'' = f, \qquad u(-\pi) = u(\pi), \qquad u'(-\pi) = u'(\pi), \tag{12.7}$$

associated with the positive semi-definite differential operator

$$K = D^* \circ D = (-D)\, D = -D^2 \tag{12.8}$$

acting on the space of 2π periodic functions. The corresponding gradient flow (12.6) is the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,-\pi) = u(t,\pi), \qquad \frac{\partial u}{\partial x}\,(t,-\pi) = \frac{\partial u}{\partial x}\,(t,\pi), \tag{12.9}$$

known as the *heat equation* since it models (among other diffusion processes) heat flow. The solution $u(t,x)$ represents the temperature at position x and time t, and turns out to be uniquely prescribed by the initial distribution:

$$u(0,x) = f(x), \qquad -\pi \le x \le \pi. \tag{12.10}$$

Heat naturally flows from hot to cold, and so the fact that it can be described by a gradient flow should not be surprising; a derivation of (12.9) from physical principles will appear in Chapter 14. Solving the periodic heat equation was the seminal problem that led Fourier to develop the profound theory that now bears his name.

As in the discrete version, the elemental solutions to a diffusion equation (12.6) are found by introducing an exponential ansatz:

$$u(t,x) = e^{-\lambda t}\, v(x), \tag{12.11}$$

in which we replace the eigenvector $\mathbf{v}$ by a function $v(x)$. These are often referred to as *separable solutions* to indicate that they are the product of a function of t alone times a function of x alone. We substitute the solution formula (12.11) into the dynamical equations (12.6). We compute

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left[\, e^{-\lambda t}\, v(x) \,\right] = -\lambda\, e^{-\lambda t}\, v(x), \quad \text{while} \quad -K[u] = -K\left[\, e^{-\lambda t}\, v(x) \,\right] = -e^{-\lambda t}\, K[v],$$

since the exponential factor is a function of t, while K only involves differentiation with respect to x. Equating these two expressions and canceling the common exponential factor, we conclude that $v(x)$ must solve a boundary value problem of the form

$$K[v] = \lambda\, v. \tag{12.12}$$

We interpret λ as the *eigenvalue* and $v(x)$ as the corresponding *eigenfunction* for the operator K subject to the relevant boundary conditions. Each eigenvalue and eigenfunction pair will produce a solution (12.11) to the partial differential equation, and the general solution can be built up through linear superposition.

For example, substitution of the exponential ansatz (12.11) into the periodic heat equation (12.9) leads to the eigenvalue problem

$$v'' + \lambda v = 0, \qquad v(-\pi) = v(\pi), \qquad v'(-\pi) = v'(\pi), \tag{12.13}$$

for the *eigenfunction* $v(x)$. Now, it is not hard to show that if $\lambda < 0$ or λ is complex, then the only periodic solution to (12.13) is the trivial solution $v(x) \equiv 0$. Thus, all eigenvalues must be real and non-negative: $\lambda \geq 0$. This is not an accident — as we will discuss in detail in Section 14.7, it is a direct consequence of the positive semi-definiteness of the underlying differential operator (12.8). When $\lambda = 0$, periodicity singles out the nonzero constants $v(x) \equiv c \neq 0$ as the associated eigenfunctions. If $\lambda = \omega^2 > 0$, then the general solution to the differential equation (12.13) is a linear combination

$$v(x) = a \cos \omega x + b \sin \omega x$$

of the basis solutions. A nonzero function of this form will satisfy the 2π periodic boundary conditions if and only if $\omega = k$ is an integer. Therefore, the eigenvalues

$$\lambda = k^2, \qquad 0 \leq k \in \mathbb{N},$$

are the squares of positive integers. Each positive eigenvalue $\lambda = k^2 > 0$ admits two linearly independent eigenfunctions, namely $\sin k x$ and $\cos k x$, while the zero eigenvalue $\lambda = 0$ has only one, the constant function 1. We conclude that the basic trigonometric functions

$$1, \qquad \cos x, \qquad \sin x, \qquad \cos 2x, \qquad \sin 2x, \qquad \cos 3x, \qquad \ldots \tag{12.14}$$

form a complete system of independent eigenfunctions for the periodic boundary value problem (12.13).

By construction, each eigenfunction gives rise to a particular solution (12.11) to the periodic heat equation (12.9). We have therefore discovered an infinite collection of independent solutions:

$$u_k(x) = e^{-k^2 t} \cos k x, \qquad \widetilde{u}_k(x) = e^{-k^2 t} \sin k x, \qquad k = 0, 1, 2, 3, \ldots .$$

Linear superposition tells us that finite linear combinations of solutions are also solutions. However, these will *not* suffice to describe the general solution to the problem, and so we are led to propose an infinite series[†]

$$u(t,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k e^{-k^2 t} \cos k x + b_k e^{-k^2 t} \sin k x \right] \tag{12.15}$$

[†] For technical reasons, one takes the basic null eigenfunction to be $\frac{1}{2}$ instead of 1. The explanation will be revealed in the following section.

to represent the general solution to the periodic heat equation. As in the discrete version, the coefficients a_k, b_k are found by substituting the solution formula into the initial condition (12.10), whence

$$u(0,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[\, a_k \cos k\,x + b_k \sin k\,x \,\right] = f(x). \tag{12.16}$$

The result is the *Fourier series* representation of the initial temperature distribution. Once we have prescribed the Fourier coefficients a_k, b_k, (12.15) provides an explicit formula for the solution to the periodic initial-boundary value problem for the heat equation.

However, since we are dealing with infinite series, the preceding is purely a formal construction, and requires some serious mathematical analysis to place it on a firm footing. The key questions are

- First, when does such an infinite trigonometric series converge?
- Second, what kinds of functions $f(x)$ can be represented by a convergent Fourier series?
- Third, if we have such an f, how do we determine its Fourier coefficients a_k, b_k?
- And lastly, since we are trying to solve differential equations, can we safely differentiate a Fourier series?

These are the fundamental questions of Fourier analysis, and must be properly dealt with before we can make any serious progress towards solving the heat equation.

A similar analysis applies to a second order dynamical system of the Newtonian form

$$\frac{\partial^2 u}{\partial t^2} = -\,K[\,u\,]. \tag{12.17}$$

Such differential equations are used to describe the free vibrations of continuous mechanical systems, such as bars, strings, and, in higher dimensions, membranes, solid bodies, fluids, etc. For example, the vibration system (12.17) corresponding to the differential operator (12.8) is the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,. \tag{12.18}$$

The wave equation models stretching vibrations of a bar, sound vibrations in a column of air, e.g., inside a wind instrument, transverse vibrations of a string, e.g., a violin string, surfaces waves on a fluid, electromagnetic waves, and a wide variety of other vibrational and wave phenomena.

As always, we need to impose suitable boundary conditions in order to proceed. Consider, for example, the wave equation with homogeneous Dirichlet boundary conditions

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,0) = 0, \qquad u(t,\ell) = 0, \tag{12.19}$$

that models, for instance, the vibrations of a uniform violin string whose ends are tied down. Adapting our discrete trigonometric ansatz, we are naturally led to look for a separable solution of the form

$$u(t,x) = \cos(\omega\,t)\,v(x) \tag{12.20}$$

in which ω represents the vibrational frequency. Substituting into the wave equation and the associated boundary conditions, we deduce that $v(x)$ must be a solution to the eigenvalue problem

$$\frac{d^2 v}{dx^2} + \omega^2 v = 0, \qquad\qquad v(0) = 0 = v(\ell), \tag{12.21}$$

in which $\omega^2 = \lambda$ plays the role of the eigenvalue. We require a *nonzero* solution to this linear boundary value problem, and this requires ω^2 to be strictly positive. As above, this can be checked by directly solving the boundary value problem, but is, in fact, a consequence of positive definiteness; see Section 14.7 for details. Assuming $\omega^2 > 0$, the general solution to the differential equation is a trigonometric function

$$v(x) = a \cos \omega\, x + b \sin \omega\, x.$$

The boundary condition at $x = 0$ requires $a = 0$, and so

$$v(x) = b \sin \omega\, x.$$

The second boundary condition requires

$$v(\ell) = b \sin \omega\, \ell = 0.$$

Assuming $b \neq 0$, as otherwise the solution is trivial, $\omega\, \ell$ must be an integer multiple of π. Thus, the natural frequencies of vibration are

$$\omega_k = \frac{k\,\pi}{\ell}\,, \qquad\qquad k = 1, 2, 3, \ldots\,.$$

The corresponding eigenfunctions are

$$v_k(x) = \sin \frac{k\,\pi\, x}{\ell}\,, \qquad\qquad k = 1, 2, 3, \ldots\,. \tag{12.22}$$

Thus, we find the following natural modes of vibration of the wave equation:

$$u_k(t, x) = \cos \frac{k\,\pi\, t}{\ell}\; \sin \frac{k\,\pi\, x}{\ell}\,, \qquad\qquad \widetilde{u}_k(t, x) = \sin \frac{k\,\pi\, t}{\ell}\; \sin \frac{k\,\pi\, x}{\ell}\,.$$

Each solution represents a spatially periodic standing wave form. We expect to write the general solution to the boundary value problem as an infinite series

$$u(t, x) = \sum_{k=1}^{\infty} \left(b_k \cos \frac{k\,\pi\, t}{\ell}\; \sin \frac{k\,\pi\, x}{\ell} + d_k \sin \frac{k\,\pi\, t}{\ell} \sin \frac{k\,\pi\, x}{\ell} \right) \tag{12.23}$$

in the natural modes. Interestingly, in this case at each fixed t, there are no cosine terms, and so we have a more specialized type of Fourier series. The same convergence issues for such *Fourier sine series* arise. It turns out that the general theory of Fourier series will also cover Fourier sine series.

We have now completed our brief introduction to the dynamical equations of continuous media and the Fourier series method of solution. The student should now be sufficiently motivated, and it is time to delve into the underlying Fourier theory. In Chapter 14 we will return to the applications to the one-dimensional heat and wave equations.

12.2. Fourier Series.

While the need to solve physically interesting partial differential equations served as our (and Fourier's) initial motivation, the remarkable range of applications qualifies Fourier's discovery as one of the most important in all of mathematics. We therefore take some time to properly develop the basic theory of Fourier series and, in the following chapter, a number of important extensions. Then, properly equipped, we will be in a position to return to the source — solving partial differential equations.

The starting point is the need to represent a given function $f(x)$, defined for $-\pi \le x \le \pi$, as a convergent series in the elementary trigonometric functions:

$$f(x) = \frac{a_0}{2} \; + \; \sum_{k=1}^{\infty} \left[\, a_k \, \cos k\,x + b_k \, \sin k\,x \,\right] . \tag{12.24}$$

The first order of business is to determine the formulae for the Fourier coefficients a_k, b_k. The key is orthogonality. We already observed, in Example 5.12, that the trigonometric functions are orthogonal with respect to the rescaled L^2 inner product

$$\langle\, f \,; g \,\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx \tag{12.25}$$

on the interval† $[\,-\pi, \pi\,]$. The explicit orthogonality relations are

$$\begin{aligned} \langle\, \cos k\,x \,; \cos l\,x \,\rangle = \langle\, \sin k\,x \,; \sin l\,x \,\rangle &= 0, && \text{for} \quad k \ne l, \\ \langle\, \cos k\,x \,; \sin l\,x \,\rangle &= 0, && \text{for all} \quad k, l, \\ \|\, 1 \,\| = \sqrt{2}\,, \qquad \|\, \cos k\,x \,\| = \|\, \sin k\,x \,\| &= 1, && \text{for} \quad k \ne 0, \end{aligned} \tag{12.26}$$

where k and l indicate non-negative integers.

Remark: If we were to replace the constant function 1 by $\frac{1}{\sqrt{2}}$, then the resulting functions would form an orthonormal system. However, this extra $\sqrt{2}$ turns out to be utterly annoying, and is best omitted from the outset.

Remark: Orthogonality of the trigonometric functions is not an accident, but follows from their status as eigenfunctions for the self-adjoint boundary value problem (12.13). The general result, to be presented in Section 14.7, is the function space analog of the orthogonality of eigenvectors of symmetric matrices, cf. Theorem 8.20.

If we ignore convergence issues for the moment, then the orthogonality relations (12.26) serve to prescribe the Fourier coefficients: Taking the inner product of both sides

† We have chosen the interval $[\,-\pi, \pi\,]$ for convenience. A common alternative is the interval $[\,0, 2\pi\,]$. In fact, since the trigonometric functions are 2π periodic, any interval of length 2π will serve equally well. Adapting Fourier series to intervals of other lengths will be discussed in Section 12.4.

with $\cos l\,x$ for $l > 0$, and invoking the underlying linearity[‡] of the inner product, yields

$$\begin{aligned}\langle\, f\,;\cos l\,x\,\rangle &= \frac{a_0}{2}\,\langle\, 1\,;\cos l\,x\,\rangle \;+\; \sum_{k=1}^{\infty}\,[\, a_k\,\langle\,\cos k\,x\,;\cos l\,x\,\rangle + b_k\,\langle\,\sin k\,x\,;\cos l\,x\,\rangle\,]\\ &= a_l\,\langle\,\cos l\,x\,;\cos l\,x\,\rangle = a_l,\end{aligned}$$

since, by the orthogonality relations (12.26), all terms but the l^{th} vanish. This serves to prescribe the Fourier coefficient a_l. A similar manipulation with $\sin l\,x$ fixes $b_l = \langle\, f\,;\sin l\,x\,\rangle$, while taking the inner product with the constant function 1 gives

$$\langle\, f\,;1\,\rangle = \frac{a_0}{2}\,\langle\, 1\,;1\,\rangle \;+\; \sum_{k=1}^{\infty}\,[\, a_k\,\langle\,\cos k\,x\,;1\,\rangle + b_k\,\langle\,\sin k\,x\,;1\,\rangle\,] = \frac{a_0}{2}\,\|\,1\,\|^2 = a_0,$$

which agrees with the preceding formula for a_l when $l = 0$, and explains why we include the extra factor of $\frac{1}{2}$ in the constant term. Thus, *if the Fourier series converges to the function $f(x)$, then its coefficients are prescribed by taking inner products with the basic trigonometric functions*. The alert reader may recognize the preceding argument — it is the function space version of our derivation or the fundamental orthonormal and orthogonal basis formulae (5.4, 7), which are valid in any inner product space. The key difference here is that we are dealing with infinite series instead of finite sums, and convergence issues must be properly addressed. However, we defer these more delicate considerations until after we have gained some basic familiarity with how Fourier series work in practice.

Let us summarize where we are with the following fundamental definition.

Definition 12.1. The *Fourier series* of a function $f(x)$ defined on $-\pi \le x \le \pi$ is the infinite trigonometric series

$$f(x) \;\sim\; \frac{a_0}{2} \;+\; \sum_{k=1}^{\infty}\,[\, a_k\,\cos k\,x + b_k\,\sin k\,x\,]\,, \tag{12.27}$$

whose coefficients are given by the inner product formulae

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos k\,x\,dx, \qquad\qquad & k &= 0,1,2,3,\ldots,\\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin k\,x\,dx, \qquad\qquad & k &= 1,2,3,\ldots. \end{aligned} \tag{12.28}$$

Note that the function $f(x)$ cannot be completely arbitrary, since, at the very least, the integrals in the coefficient formulae must be well defined and finite. Even if the coefficients (12.28) are finite, there is no guarantee that the resulting Fourier series converges, and, even if it converges, no guarantee that it converges to the original function $f(x)$. For these reasons, we use the $\sim$ symbol instead of an equals sign when writing down a Fourier series. Before tackling these key issues, let us look at an elementary example.

[‡] More rigorously, linearity only applies to finite linear combinations, not infinite series. Here, thought, we are just trying to establish and motivate the basic formulae, and can safely defer such technical complications until the final section.

Example 12.2. Consider the function $f(x) = x$. We may compute its Fourier coefficients directly, employing integration by parts to evaluate the integrals:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\, dx = 0, \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos k\,x\, dx = \frac{1}{\pi}\left[\, \frac{x\,\sin k\,x}{k} + \frac{\cos k\,x}{k^2}\,\right]\Bigg|_{x=-\pi}^{\pi} = 0,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin k\,x\, dx = \frac{1}{\pi}\left[\, -\frac{x\,\cos k\,x}{k} + \frac{\sin k\,x}{k^2}\,\right]\Bigg|_{x=-\pi}^{\pi} = \frac{2}{k}\,(-1)^{k+1}\,. \tag{12.29}$$

Therefore, the Fourier cosine coefficients of the function x all vanish, $a_k = 0$, and its Fourier series is

$$x \;\sim\; 2\left(\sin x \;-\; \frac{\sin 2\,x}{2} \;+\; \frac{\sin 3\,x}{3} \;-\; \frac{\sin 4\,x}{4} \;+\; \cdots\right). \tag{12.30}$$

Convergence of this series is *not* an elementary matter. Standard tests, including the ratio and root tests, fail to apply. Even if we know that the series converges (which it does — for all x), it is certainly not obvious what function it converges to. Indeed, it *cannot* converge to the function $f(x) = x$ for all values of x. If we substitute $x = \pi$, then every term in the series is zero, and so the Fourier series converges to 0 — which is not the same as $f(\pi) = \pi$.

The n^{th} *partial sum* of a Fourier series is the trigonometric polynomial†

$$s_n(x) \;=\; \frac{a_0}{2} + \sum_{k=1}^{n} \left[\, a_k\, \cos k\,x + b_k\, \sin k\,x\,\right]. \tag{12.31}$$

By definition, the Fourier series *converges* at a point x if and only if the partial sums have a limit:

$$\lim_{n\to\infty} s_n(x) = \widetilde{f}(x), \tag{12.32}$$

which may or may not equal the value of the original function $f(x)$. Thus, a key requirement is to formulate conditions on the function $f(x)$ that guarantee that the Fourier series converges, and, even more importantly, the limiting sum reproduces the original function: $\widetilde{f}(x) = f(x)$. This will all be done in detail below.

Remark: The passage from trigonometric polynomials to Fourier series is similar to the passage from polynomials to power series. A power series

$$f(x) \;\sim\; c_0 + c_1\, x + \;\cdots\; + c_n\, x^n + \;\cdots\; = \;\sum_{k=0}^{\infty} c_k\, x^k$$

can be viewed as an infinite linear combination of the basic monomials $1, x, x^2, x^3, \ldots$. According to Taylor's formula, (C.3), the coefficients $c_k = \dfrac{f^{(k)}(0)}{k!}$ are given in terms of

† The reason for the term "trigonometric polynomial" was discussed at length in Example 2.17(c).

the derivatives of the function at the origin. The partial sums

$$s_n(x) = c_0 + c_1\,x + \;\cdots\; + c_n\,x^n \;=\; \sum_{k=0}^{n} c_k\,x^k$$

of a power series are ordinary polynomials, and the same convergence issues arise.

Although superficially similar, in actuality the two theories are profoundly different. Indeed, while the theory of power series was well established in the early days of the calculus, there remain, to this day, unresolved foundational issues in Fourier theory. A power series either converges everywhere, or on an interval centered at 0, or nowhere except at 0. (See Section 16.2 for additional details.) On the other hand, a Fourier series can converge on quite bizarre sets. In fact, the detailed analysis of the convergence properties of Fourier series led the nineteenth century German mathematician Georg Cantor to formulate modern set theory, and, thus, played a seminal role in the establishment of the foundations of modern mathematics. Secondly, when a power series converges, it converges to an analytic function, which is infinitely differentiable, and whose derivatives are represented by the power series obtained by termwise differentiation. Fourier series may converge, not only to periodic continuous functions, but also to a wide variety of discontinuous functions and, even, when suitably interpreted, to generalized functions like the delta function! Therefore, the termwise differentiation of a Fourier series is a nontrivial issue.

Once one comprehends how different the two subjects are, one begins to understand why Fourier's astonishing claims were initially widely disbelieved. Before the advent of Fourier, mathematicians only accepted analytic functions as the genuine article. The fact that Fourier series can converge to nonanalytic, even discontinuous functions was extremely disconcerting, and resulted in a complete re-evaluation of function theory, culminating in the modern definition of function that you now learn in first year calculus. Only through the combined efforts of many of the leading mathematicians of the nineteenth century was a rigorous theory of Fourier series firmly established; see Section 12.5 for the main details and the advanced text [**199**] for a comprehensive treatment.

Periodic Extensions

The trigonometric constituents (12.14) of a Fourier series are all periodic functions of period 2π. Therefore, if the series converges, the limiting function $\widetilde{f}(x)$ must also be periodic of period 2π:

$$\widetilde{f}(x + 2\pi) = \widetilde{f}(x) \qquad \text{for all} \qquad x \in \mathbb{R}.$$

A Fourier series can only converge to a 2π periodic function. So it was unreasonable to expect the Fourier series (12.30) to converge to the non-periodic to $f(x) = x$ everywhere. Rather, it should converge to its periodic extension, as we now define.

Lemma 12.3. *If $f(x)$ is any function defined for $-\pi < x \le \pi$, then there is a unique 2π periodic function $\widetilde{f}$, known as the 2π* periodic extension *of f, that satisfies $\widetilde{f}(x) = f(x)$ for all $-\pi < x \le \pi$.*

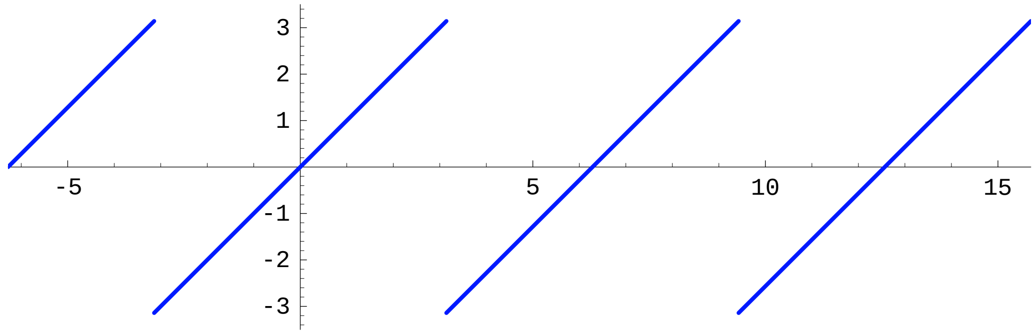

Figure 12.1. Periodic extension of x.

Proof: Pictorially, the graph of the periodic extension of a function $f(x)$ is obtained by repeatedly copying that part of the graph of f between $-\pi$ and π to adjacent intervals of length 2π; Figure 12.1 shows a simple example. More formally, given $x \in \mathbb{R}$, there is a unique integer m so that $(2m-1)\pi < x \leq (2m+1)\pi$. Periodicity of $\widetilde{f}$ leads us to define

$$\widetilde{f}(x) = \widetilde{f}(x - 2m\pi) = f(x - 2m\pi), \tag{12.33}$$

noting that if $-\pi < x \leq \pi$, then $m = 0$ and hence $\widetilde{f}(x) = f(x)$ for such x. The proof that the resulting function $\widetilde{f}$ is 2π periodic is left as Exercise ■. *Q.E.D.*

Remark: The construction of the periodic extension of Lemma 12.3 uses the value $f(\pi)$ at the right endpoint and requires $\widetilde{f}(-\pi) = \widetilde{f}(\pi) = f(\pi)$. One could, alternatively, require $\widetilde{f}(\pi) = \widetilde{f}(-\pi) = f(-\pi)$, which, if $f(-\pi) \neq f(\pi)$, leads to a slightly different 2π periodic extension of the function. There is no *a priori* reason to prefer one over the other. In fact, for Fourier theory, as we shall discover, one should use neither, but rather an "average" of the two. Thus, the preferred Fourier periodic extension $\widetilde{f}(x)$ will satisfy

$$\widetilde{f}(\pi) = \widetilde{f}(-\pi) = \tfrac{1}{2}\left[\, f(\pi) + f(-\pi) \,\right], \tag{12.34}$$

which then fixes its values at the odd multiples of π.

Example 12.4. The 2π periodic extension $\widetilde{f}(x)$ of $f(x) = x$ is the "sawtooth" function graphed in Figure 12.1. It agrees with x between $-\pi$ and π. Since $f(\pi) = \pi$, $f(-\pi) = -\pi$, the Fourier extension (12.34) sets $\widetilde{f}(k\pi) = 0$ for any odd integer k. Explicitly,

$$\widetilde{f}(x) = \begin{cases} x - 2m\pi, & (2m-1)\pi < x < (2m+1)\pi, \\ 0, & x = (2m-1)\pi, \end{cases} \qquad \text{where } m \text{ is any integer.}$$

With this convention, it can be proved that the Fourier series (12.30) converges everywhere to the 2π periodic extension $\widetilde{f}(x)$. In particular,

$$2 \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\sin kx}{k} = \begin{cases} x, & -\pi < x < \pi, \\ 0, & x = \pm\pi. \end{cases} \tag{12.35}$$

Even this very simple example has remarkable and nontrivial consequences. For instance, if we substitute $x = \frac{1}{2}\pi$ in (12.30) and divide by 2, we obtain *Gregory's series*

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots . \tag{12.36}$$

While this striking formula predates Fourier theory — it was, in fact, first discovered by Leibniz — a direct proof is not easy.

Remark: While numerologically fascinating, Gregory's series is of scant practical use for actually computing π since its rate of convergence is painfully slow. The reader may wish to try adding up terms to see how far out one needs to go to accurately compute even the first two decimal digits of π. Round-off errors will eventually interfere with any attempt to compute the complete summation to any reasonable degree of accuracy.

Piecewise Continuous Functions

As we shall see, all continuously differentiable, 2π periodic functions can be represented as convergent Fourier series. More generally, we can allow the function to have some simple discontinuities. Although not the most general class of functions that possess convergent Fourier series, such "piecewise continuous" functions will suffice for all the applications we consider in this text.

Definition 12.5. A function $f(x)$ is said to be *piecewise continuous* on an interval $[a,b]$ if it is defined and continuous except possibly at a finite number of points $a \le x_1 < x_2 < \ldots < x_n \le b$. At each point of discontinuity, the left and right hand limits†

$$f(x_k^-) = \lim_{x \to x_k^-} f(x), \qquad\qquad f(x_k^+) = \lim_{x \to x_k^+} f(x),$$

exist. Note that we do not require that $f(x)$ be defined at x_k. Even if $f(x_k)$ is defined, it does not necessarily equal either the left or the right hand limit.

A function $f(x)$ defined for all $x \in \mathbb{R}$ is piecewise continuous provided it is piecewise continuous on every bounded interval. In particular, a 2π periodic function $\widetilde{f}(x)$ is piecewise continuous if and only if it is piecewise continuous on the interval $[-\pi,\pi]$.

A representative graph of a piecewise continuous function appears in Figure 12.2. The points x_k are known as *jump discontinuities* of $f(x)$ and the difference

$$\beta_k = f(x_k^+) - f(x_k^-) = \lim_{x \to x_k^+} f(x) - \lim_{x \to x_k^-} f(x) \tag{12.37}$$

between the left and right hand limits is the *magnitude* of the jump, cf. (11.49). If $\beta_k = 0$, and so the right and left hand limits agree, then the discontinuity is *removable* since redefining $f(x_k) = f(x_k^+) = f(x_k^-)$ makes f continuous at x_k. We will assume, without significant loss of generality, that our functions have no removable discontinuities.

† At the endpoints a, b we only require one of the limits, namely $f(a^+)$ and $f(b^-)$, to exist.

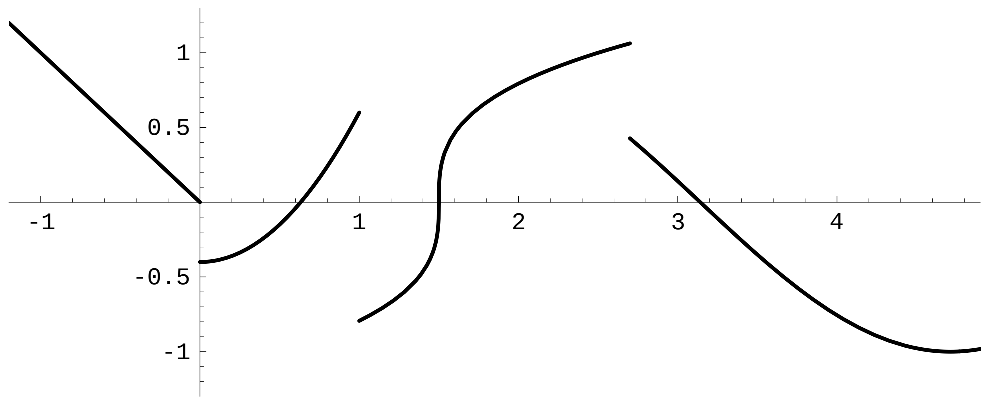

Figure 12.2. Piecewise Continuous Function.

The simplest example of a piecewise continuous function is the step function

$$\sigma(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{12.38}$$

It has a single jump discontinuity at $x = 0$ of magnitude 1, and is continuous — indeed, constant — everywhere else. If we translate and scale the step function, we obtain a function

$$h(x) = \beta\,\sigma(x-y) = \begin{cases} \beta, & x > y, \\ 0, & x < y, \end{cases} \tag{12.39}$$

with a single jump discontinuity of magnitude β at the point $x = y$.

If $f(x)$ is any piecewise continuous function, then its Fourier coefficients are well-defined — the integrals (12.28) exist and are finite. Continuity, however, is not enough to ensure convergence of the resulting Fourier series.

Definition 12.6. A function $f(x)$ is called *piecewise* C^1 on an interval $[a,b]$ if it is defined, continuous and continuously differentiable except possibly at a finite number of points $a \leq x_1 < x_2 < \ldots < x_n \leq b$. At each exceptional point, the left and right hand limits† exist:

$$f(x_k^-) = \lim_{x \to x_k^-} f(x), \qquad f(x_k^+) = \lim_{x \to x_k^+} f(x),$$
$$f'(x_k^-) = \lim_{x \to x_k^-} f'(x), \qquad f'(x_k^+) = \lim_{x \to x_k^+} f'(x).$$

See Figure 12.3 for a representative graph. For a piecewise continuous C^1 function, an exceptional point x_k is either

- a *jump discontinuity* of f, but where the left and right hand derivatives exist, or
- a *corner*, meaning a point where f is continuous, so $f(x_k^-) = f(x_k^+)$, but has different left and right hand derivatives: $f'(x_k^-) \neq f'(x_k^+)$.

† As before, at the endpoints we only require the appropriate one-sided limits, namely $f(a^+)$, $f'(a^+)$ and $f(b^-)$, $f'(b^-)$, to exist.

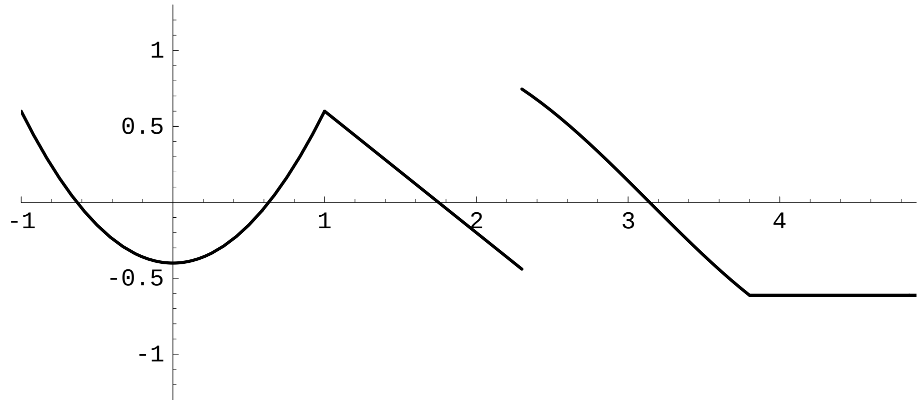

Figure 12.3. Piecewise C^1 Function.

Thus, at each point, including jump discontinuities, the graph of $f(x)$ has well-defined right and left tangent lines. For example, the function $f(x) = |\,x\,|$ is piecewise C^1 since it is continuous everywhere and has a corner at $x = 0$, with $f'(0^+) = +1$, $f'(0^-) = -1$.

There is an analogous definition of a piecewise C^n function. One requires that the function has n continuous derivatives, except at a finite number of points. Moreover, at every point, the function has well-defined right and left hand limits of all its derivatives up to order n.

The Convergence Theorem

We are now able to state the fundamental convergence theorem for Fourier series.

Theorem 12.7. *If $\widetilde{f}(x)$ is any 2π periodic, piecewise C^1 function, then, for any $x \in \mathbb{R}$, its Fourier series converges to*

$$\begin{array}{ll} \widetilde{f}(x), & \text{if } \widetilde{f} \text{ is continuous at } x, \\ \frac{1}{2}\left[\,\widetilde{f}(x^+) + \widetilde{f}(x^-)\,\right], & \text{if } x \text{ is a jump discontinuity.} \end{array}$$

Thus, the Fourier series converges, as expected, to $\widetilde{f}(x)$ at all points of continuity; at discontinuities, the Fourier series can't decide whether to converge to the right or left hand limit, and so ends up "splitting the difference" by converging to their average; see Figure 12.4. If we redefine $\widetilde{f}(x)$ at its jump discontinuities to have the average limiting value, so

$$\widetilde{f}(x) = \tfrac{1}{2}\left[\,\widetilde{f}(x^+) + \widetilde{f}(x^-)\,\right], \tag{12.40}$$

— an equation that automatically holds at all points of continuity — then Theorem 12.7 would say that the Fourier series converges to $\widetilde{f}(x)$ everywhere. We will discuss the ideas underlying the proof of the Convergence Theorem 12.7 at the end of Section 12.5.

Example 12.8. Let $\sigma(x)$ denote the step function (12.38). Its Fourier coefficients

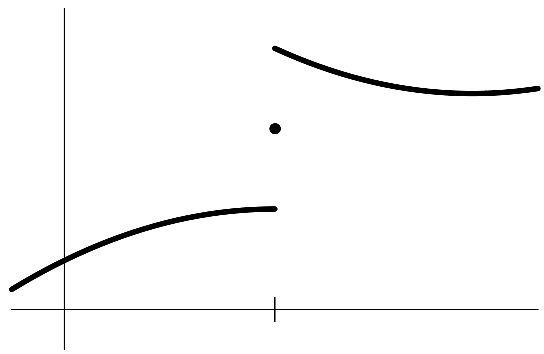

Figure 12.4. Splitting the Difference.

are easily computed:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi} dx = 1,$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma(x)\cos k\,x\,dx = \frac{1}{\pi}\int_{0}^{\pi} \cos k\,x\,dx = 0,$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} \sigma(x)\sin k\,x\,dx = \frac{1}{\pi}\int_{0}^{\pi} \sin k\,x\,dx = \begin{cases} \dfrac{2}{k\,\pi}, & k = 2\,l+1 \text{ odd}, \\ 0, & k = 2\,l \text{ even}. \end{cases}$$

Therefore, the Fourier series for the step function is

$$\sigma(x) \sim \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{\sin 3\,x}{3} + \frac{\sin 5\,x}{5} + \frac{\sin 7\,x}{7} + \cdots \right). \qquad (12.41)$$

According to Theorem 12.7, the Fourier series will converge to the 2π periodic extension of the step function:

$$\widetilde{\sigma}(x) = \begin{cases} 0, & (2m-1)\,\pi < x < 2\,m\pi, \\ 1, & 2\,m\pi < x < (2\,m+1)\,\pi, \\ \frac{1}{2}, & x = m\pi, \end{cases} \qquad \text{where } m \text{ is any integer},$$

which is plotted in Figure 12.5. Observe that, in accordance with Theorem 12.7, $\widetilde{\sigma}(x)$ takes the midpoint value $\frac{1}{2}$ at the jump discontinuities $0, \pm\pi, \pm 2\pi, \ldots$.

It is instructive to investigate the convergence of this particular Fourier series in some detail. Figure 12.6 displays a graph of the first few partial sums, taking, respectively, $n = 3, 5$, and 10 terms. The reader will notice that away from the discontinuities, the series does appear to be converging, albeit slowly. However, near the jumps there is a consistent overshoot of about 9%. The region where the overshoot occurs becomes narrower and narrower as the number of terms increases, but the magnitude of the overshoot persists no matter how many terms are summed up. This was first noted by the American physicist Josiah Gibbs, and is now known as the *Gibbs phenomenon* in his honor. The Gibbs overshoot is a manifestation of the subtle non-uniform convergence of the Fourier series.

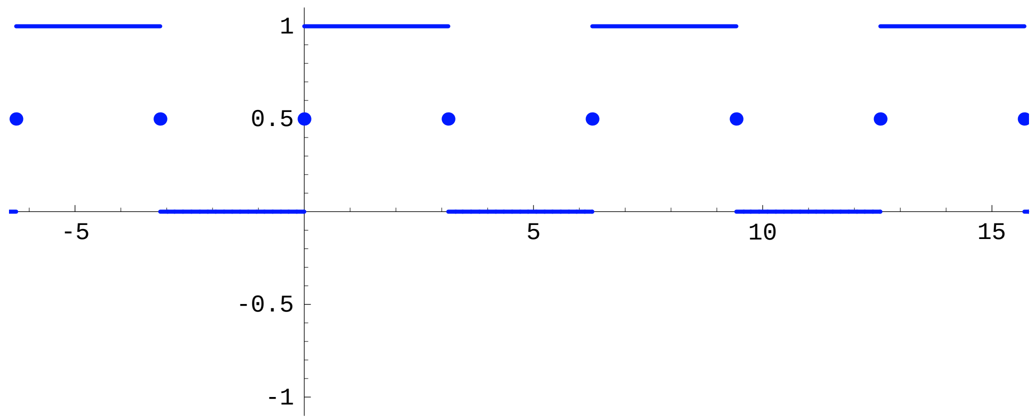

Figure 12.5. Periodic Step Function.

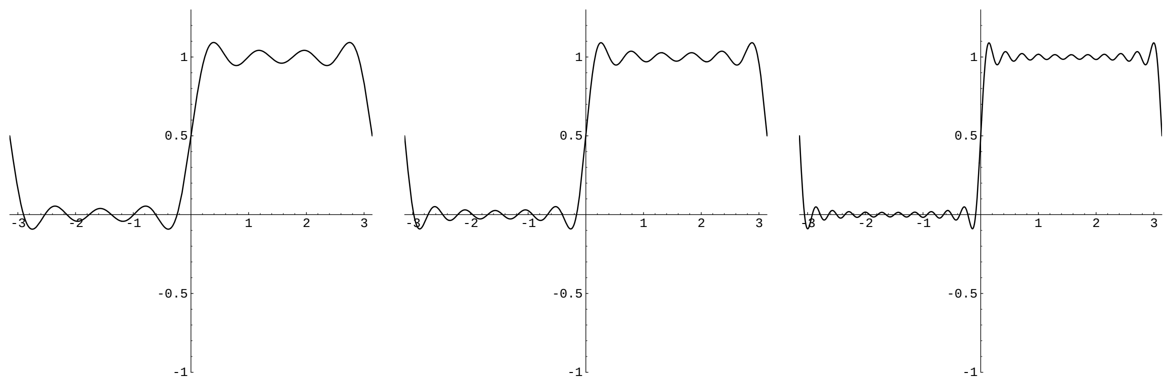

Figure 12.6. Gibbs Phenomenon.

Even and Odd Functions

We already noted that the Fourier cosine coefficients of the function $f(x) = x$ are all 0. This is not an accident, but rather a direct consequence of the fact that x is an odd function. Recall first the basic definition:

Definition 12.9. A function is called *even* if $f(-x) = f(x)$. A function is *odd* if $f(-x) = -f(x)$.

For example, the functions 1, $\cos kx$, and x^2 are all even, whereas x, $\sin kx$, and $\operatorname{sign} x$ are odd. We require two elementary lemmas, whose proofs are left to the reader.

Lemma 12.10. *The sum, $f(x) + g(x)$, of two even functions is even; the sum of two odd functions is odd. The product $f(x)\,g(x)$ of two even functions, or of two odd functions, is an even function. The product of an even and an odd function is odd.*

Remark: *Every* function can be represented as the sum of an even and an odd function; see Exercise ■.

Lemma 12.11. *If $f(x)$ is odd and integrable on the symmetric interval $[-a,a]$, then* $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$. *If $f(x)$ is even and integrable, then* $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

The next result is an immediate consequence of applying Lemmas 12.10 and 12.11 to the Fourier integrals (12.28).

Proposition 12.12. *If $f(x)$ is even, then its Fourier sine coefficients all vanish, $b_k = 0$, and so f can be represented by a* Fourier cosine series

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\,x\,, \tag{12.42}$$

where

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cos k\,x\,dx, \qquad k = 0,1,2,3,\ldots\,. \tag{12.43}$$

If $f(x)$ is odd, then its Fourier cosine coefficients vanish, $a_k = 0$, and so f can be represented by a Fourier sine series

$$f(x) \sim \sum_{k=1}^{\infty} b_k \sin k\,x\,, \tag{12.44}$$

where

$$b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin k\,x\,dx, \qquad k = 1,2,3,\ldots\,. \tag{12.45}$$

Conversely, a convergent Fourier cosine (respectively, sine) series always represents an even (respectively, odd) function.

Example 12.13. The absolute value $f(x) = |\,x\,|$ is an even function, and hence has a Fourier cosine series. The coefficients are

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi, \tag{12.46}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} x\cos k\,x\,dx = \frac{2}{\pi}\left[\,\frac{x\sin k\,x}{k} + \frac{\cos k\,x}{k^2}\,\right]_{x=0}^{\pi} = \begin{cases} 0, & 0 \neq k \text{ even},\\ -\dfrac{4}{k^2\,\pi}\,, & k \text{ odd}. \end{cases}$$

Therefore

$$|\,x\,| \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3\,x}{9} + \frac{\cos 5\,x}{25} + \frac{\cos 7\,x}{49} + \cdots\right). \tag{12.47}$$

According to Theorem 12.7, this Fourier cosine series converges to the 2π periodic extension of $|\,x\,|$, the "sawtooth function" graphed in Figure 12.7.

In particular, if we substitute $x = 0$, we obtain another interesting series

$$\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2\,n+1)^2}\,. \tag{12.48}$$

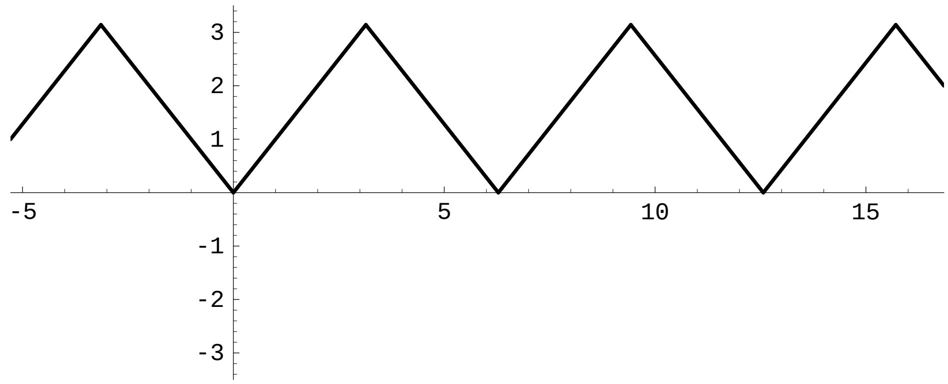

Figure 12.7. Periodic extension of $|x|$.

It converges faster than Gregory's series (12.36), and, while far from optimal in this regards, can be used to compute reasonable approximations to π. One can further manipulate this result to compute the sum of the series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \cdots .$$

We note that

$$\frac{S}{4} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \cdots .$$

Therefore, by (12.48),

$$\frac{3}{4} S = S - \frac{S}{4} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8},$$

from which we conclude that

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}. \tag{12.49}$$

Remark: The most famous function in number theory — and the source of the most outstanding problem in mathematics, the Riemann hypothesis — is the *Riemann zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{12.50}$$

Formula (12.49) shows that $\zeta(2) = \frac{1}{6}\pi^2$. In fact, the value of the zeta function at any *even* positive integer $s = 2n$ can be written as a rational polynomial in π.

If $f(x)$ is any function defined on $[0, \pi]$, then its *Fourier cosine series* is defined by the formulas (12.42–43); the resulting series represents its even, 2π periodic extension. For example, the cosine series of $f(x) = x$ is given in (12.47); indeed the even, 2π periodic extension of x coincides with the 2π periodic extension of $|x|$. Similarly, the formulas (12.44, 45) define its *Fourier sine series*, representing its odd, 2π periodic extension. In particular, since $f(x) = x$ is already odd, its Fourier sine series concides with its ordinary Fourier series.

Complex Fourier Series

An alternative, and often more convenient, approach to Fourier series is to use complex exponentials instead of sines and cosines. Indeed, Euler's formula

$$e^{\mathrm{i}\,k x} = \cos k x + \mathrm{i}\, \sin k x, \qquad e^{-\mathrm{i}\,k x} = \cos k x - \mathrm{i}\, \sin k x, \tag{12.51}$$

shows how to write the trigonometric functions

$$\cos k x = \frac{e^{\mathrm{i}\,k x} + e^{-\mathrm{i}\,k x}}{2}, \qquad \sin k x = \frac{e^{\mathrm{i}\,k x} - e^{-\mathrm{i}\,k x}}{2\,\mathrm{i}}, \tag{12.52}$$

in terms of complex exponentials. Orthonormality with respect to the rescaled L^2 Hermitian inner product

$$\langle f ; g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx\,, \tag{12.53}$$

was proved by direct computation in Example 3.45:

$$\begin{aligned} \langle e^{\mathrm{i}\,k x} ; e^{\mathrm{i}\,l x} \rangle &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\mathrm{i}\,(k-l) x}\, dx = \begin{cases} 1, & k = l, \\ 0, & k \neq l, \end{cases} \\ \| e^{\mathrm{i}\,k x} \|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} | e^{\mathrm{i}\,k x} |^2\, dx = 1. \end{aligned} \tag{12.54}$$

Again, orthogonality follows from their status as (complex) eigenfunctions for the periodic boundary value problem (12.13).

The *complex Fourier series* for a (piecewise continuous) real or complex function f is

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k\, e^{\mathrm{i}\,k x} = \cdots + c_{-2}\, e^{-2\,\mathrm{i}\,x} + c_{-1}\, e^{-\mathrm{i}\,x} + c_0 + c_1\, e^{\mathrm{i}\,x} + c_2\, e^{2\,\mathrm{i}\,x} + \cdots. \tag{12.55}$$

The orthonormality formulae (12.53) imply that the *complex Fourier coefficients* are obtained by taking the inner products

$$c_k = \langle f ; e^{\mathrm{i}\,k x} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-\mathrm{i}\,k x}\, dx \tag{12.56}$$

with the associated complex exponential. Pay attention to the minus sign in the integrated exponential — the result of taking the complex conjugate of the second argument in the inner product (12.53). It should be emphasized that the real (12.27) and complex (12.55) Fourier formulae are just two different ways of writing the *same* series! Indeed, if we apply

Euler's formula (12.51) to (12.56) and compare with the real Fourier formulae (12.28), we find that the real and complex Fourier coefficients are related by

$$
\begin{aligned}
a_k &= c_k + c_{-k}, \qquad & c_k &= \tfrac{1}{2}(a_k - \mathrm{i}\, b_k), \\
b_k &= \mathrm{i}\,(c_k - c_{-k}), \qquad & c_{-k} &= \tfrac{1}{2}(a_k + \mathrm{i}\, b_k),
\end{aligned}
\qquad k = 0, 1, 2, \ldots . \tag{12.57}
$$

Remark: We already see one advantage of the complex version. The constant function $1 = e^{0\,\mathrm{i}\,x}$ no longer plays an anomalous role — the annoying factor of $\frac{1}{2}$ in the real Fourier series (12.27) has mysteriously disappeared!

Example 12.14. For the step function $\sigma(x)$ considered in Example 12.8, the complex Fourier coefficients are

$$
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma(x)\, e^{-\mathrm{i}\,k\,x}\, dx = \frac{1}{2\pi} \int_{0}^{\pi} e^{-\mathrm{i}\,k\,x}\, dx = \begin{cases} \frac{1}{2}, & k = 0, \\ 0, & 0 \neq k \text{ even}, \\ \dfrac{1}{\mathrm{i}\,k\,\pi}\,, & k \text{ odd}. \end{cases}
$$

Therefore, the step function has the complex Fourier series

$$
\sigma(x) \;\sim\; \frac{1}{2} \;-\; \frac{\mathrm{i}}{\pi} \sum_{l=-\infty}^{\infty} \frac{e^{(2\,l+1)\,\mathrm{i}\,x}}{2\,l+1}\,.
$$

You should convince yourself that this is *exactly the same series* as the real Fourier series (12.41). We are merely rewriting it using complex exponentials instead of real sines and cosines.

Example 12.15. Let us find the Fourier series for the exponential function $e^{a\,x}$. It is much easier to evaluate the integrals for the complex Fourier coefficients, and so

$$
\begin{aligned}
c_k &= \langle\, e^{a\,x}\,; e^{\mathrm{i}\,k\,x}\,\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-\mathrm{i}\,k)\,x}\, dx = \left. \frac{e^{(a-\mathrm{i}\,k)\,x}}{2\pi\,(a-\mathrm{i}\,k)} \right|_{x=-\pi}^{\pi} \\
&= \frac{e^{(a-\mathrm{i}\,k)\,\pi} - e^{-(a-\mathrm{i}\,k)\,\pi}}{2\pi\,(a-\mathrm{i}\,k)} = (-1)^k\, \frac{e^{a\,\pi} - e^{-a\,\pi}}{2\pi\,(a-\mathrm{i}\,k)} = \frac{(-1)^k (a+\mathrm{i}\,k)\,\sinh a\,\pi}{\pi\,(a^2+k^2)}\,.
\end{aligned}
$$

Therefore, the desired Fourier series is

$$
e^{a\,x} \;\sim\; \frac{\sinh a\,\pi}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k (a+\mathrm{i}\,k)}{a^2+k^2}\, e^{\mathrm{i}\,k\,x}. \tag{12.58}
$$

As an exercise, the reader should try writing this as a real Fourier series, either by breaking up the complex series into its real and imaginary parts, or by direct evaluation of the real coefficients via their integral formulae (12.28). According to Theorem 12.7 (which is equally valid for complex Fourier series) the Fourier series converges to the 2π periodic extension of the exponential function, graphed in Figure 12.8.

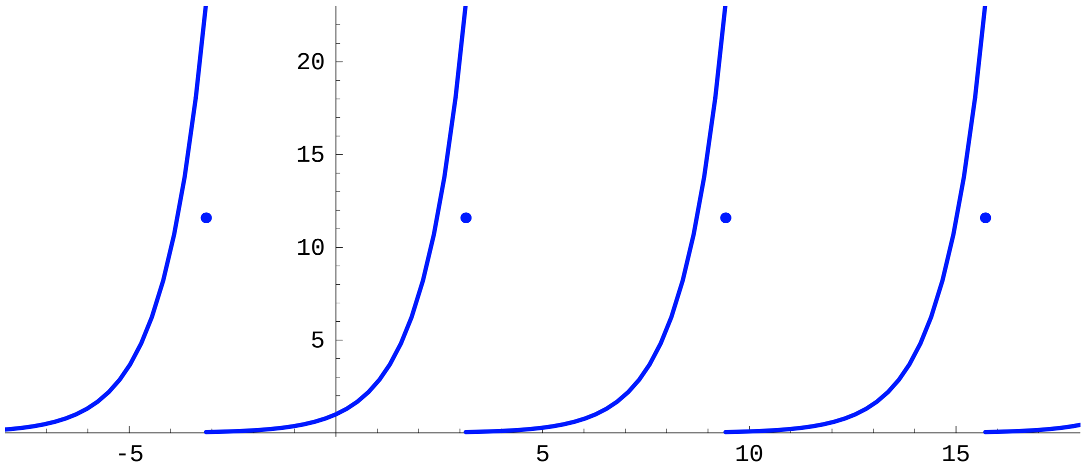

Figure 12.8. Periodic Extension of e^x.

The Delta Function

Fourier series can even be used to represent more general objects than mere functions. The most important example is the delta function $\delta(x)$. Using its characterizing properties (11.37), the real Fourier coefficients are computed as

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\cos k\,x\,dx = \frac{1}{\pi}\cos k\,0 = \frac{1}{\pi}\,, \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \delta(x)\sin k\,x\,dx = \frac{1}{\pi}\sin k\,0 = 0. \end{aligned} \tag{12.59}$$

Therefore,

$$\delta(x) \;\sim\; \frac{1}{2\pi} \;+\; \frac{1}{\pi}\left(\cos x + \cos 2x + \cos 3x + \;\cdots\;\right). \tag{12.60}$$

Since $\delta(x)$ is an even function, it should come as no surprise that it has a cosine series.

To understand in what sense this series converges to the delta function, it will help to rewrite it in complex form

$$\delta(x) \;\sim\; \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{\,i\,k\,x} = \frac{1}{2\pi}\left(\;\cdots\; + e^{-2\,i\,x} + e^{-\,i\,x} + 1 + e^{\,i\,x} + e^{2\,i\,x} + \;\cdots\;\right). \tag{12.61}$$

where the complex Fourier coefficients are computed[†] as

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \delta(x)\,e^{-\,i\,k\,x}\,dx = \frac{1}{2\pi}\,.$$

[†] Or, we could use (12.57).

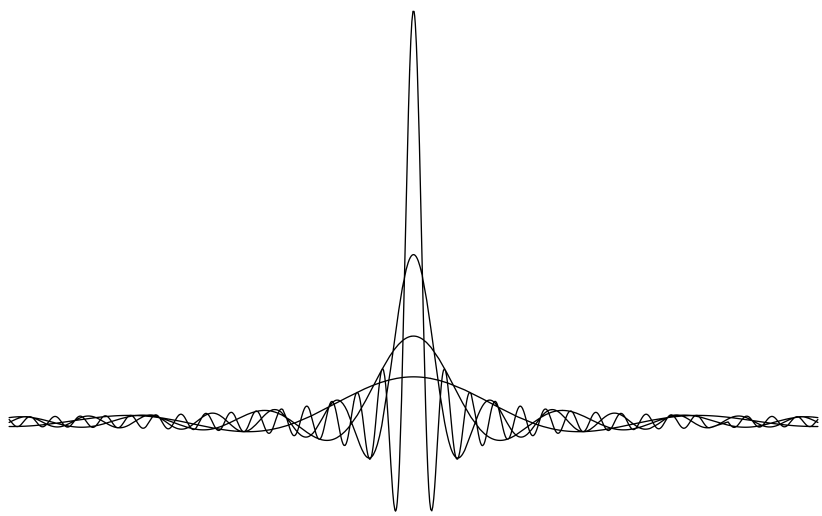

Figure 12.9. Partial Fourier Sums Approximating the Delta Function.

The n^{th} partial sum

$$s_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{\,i\,k\,x} = \frac{1}{2\pi} \left(e^{-\,i\,n\,x} + \;\cdots\; + e^{-\,i\,x} + 1 + e^{\,i\,x} + \;\cdots\; + e^{\,i\,n\,x} \right)$$

can, in fact, be explicitly evaluated. Recall the formula for the sum of a geometric series

$$\sum_{k=0}^{m} a\,r^k = a + a\,r + a\,r^2 + \;\cdots\; + a\,r^m = a \left(\frac{r^{m+1} - 1}{r - 1} \right). \tag{12.62}$$

The partial sum $s_n(x)$ has this form, with $m+1 = 2n+1$ summands, initial term $a = e^{-\,i\,n x}$, and ratio $r = e^{\,i\,x}$. Therefore,

$$\begin{aligned} s_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{\,i\,k\,x} &= \frac{1}{2\pi}\, e^{-\,i\,n x} \left(\frac{e^{\,i\,(2n+1)x} - 1}{e^{\,i\,x} - 1} \right) = \frac{1}{2\pi}\, \frac{e^{\,i\,(n+1)x} - e^{-\,i\,n x}}{e^{\,i\,x} - 1} \\ &= \frac{1}{2\pi}\, \frac{e^{\,i\left(n+\frac{1}{2}\right)x} - e^{-\,i\left(n+\frac{1}{2}\right)x}}{e^{\,i\,x/2} - e^{-\,i\,x/2}} = \frac{1}{2\pi}\, \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\,. \end{aligned} \tag{12.63}$$

In this computation, to pass from the first to the second line, we multiplied numerator and denominator by $e^{-\,i\,x/2}$, after which we used the formula (3.86) for the sine function in terms of complex exponentials. Incidentally, (12.63) is equivalent to the intriguing trigonometric summation formula

$$s_n(x) = \frac{1}{2\pi} + \frac{1}{\pi} \left(\cos x + \cos 2x + \cos 3x + \;\cdots\; + \cos n\,x \right) = \frac{1}{2\pi}\, \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\,. \tag{12.64}$$

Graphs of the partial sums $s_n(x)$ for several values of n are displayed in Figure 12.9. Note that the spike, at $x = 0$, progressively becomes taller and thinner, converging to an infinitely tall, infinitely thin delta spike. Indeed, by l'Hôpital's Rule,

$$\lim_{x\to 0}\ \frac{1}{2\pi}\,\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x} = \lim_{x\to 0}\ \frac{1}{2\pi}\,\frac{\left(n+\frac{1}{2}\right)\cos\left(n+\frac{1}{2}\right)x}{\frac{1}{2}\cos\frac{1}{2}x} = \frac{n+\frac{1}{2}}{\pi}\ \longrightarrow\ \infty \quad \text{as} \quad n\to\infty.$$

(An elementary proof of this formula is to note that, at $x = 0$, every term in the original sum (12.64) is equal to 1.) Furthermore, the integrals remain fixed,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} s_n(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{\,\mathrm{i}\,k\,x}\,dx = 1, \tag{12.65}$$

as required for convergence to the delta function. However, away from the spike, the partial sums do *not* go to zero! Rather, they oscillate more and more rapidly, maintaining an overall amplitude of $\frac{1}{2\pi}\csc\frac{1}{2}x = 1/\left(2\pi\sin\frac{1}{2}x\right)$. As n gets large, the amplitude function appears as an envelope of the increasingly rapid oscillations. Roughly speaking, the fact that $s_n(x)\to\delta(x)$ as $n\to\infty$ means that the "infinitely fast" oscillations somehow cancel each other out, and the net effect is zero away from the spike at $x = 0$. Thus, the convergence of the Fourier sums to $\delta(x)$ is much more subtle than in the original limiting definition (11.31). The technical term is *weak* convergence, which plays an very important role in advanced mathematical analysis, [**159**]; see Exercise ■ below for additional details.

Remark: Although we stated that the Fourier series (12.60, 61) represent the delta function, this is not entirely correct. Remember that a Fourier series converges to the 2π periodic extension of the original function. Therefore, (12.61) actually represents the periodic extension of the delta function:

$$\widetilde{\delta}(x) = \ \cdots + \delta(x+4\pi)+\delta(x+2\pi)+\delta(x)+\delta(x-2\pi)+\delta(x-4\pi)+\delta(x-6\pi)+\ \cdots\ , \tag{12.66}$$

consisting of a periodic array of delta spikes concentrated at all integer multiples of 2π.

12.3. Differentiation and Integration.

If a series of functions converges "nicely" then one expects to be able to integrate and differentiate it term by term; the resulting series should converge to the integral and derivative of the original sum. Integration and differentiation of power series is always valid within the range of convergence, and is used extensively in the construction of series solutions of differential equations, series for integrals of non-elementary functions, and so on. The interested reader can consult Appendix C for further details.

As we now appreciate, the convergence of Fourier series is a much more delicate matter, and so one must be considerably more careful with their differentiation and integration. Nevertheless, in favorable situations, both operations lead to valid results, and are quite useful for constructing Fourier series of more complicated functions. It is a remarkable, profound fact that Fourier analysis is completely compatible with the calculus of generalized functions that we developed in Chapter 11. For instance, differentiating the Fourier series for a piecewise C^1 function leads to the Fourier series for the differentiated function that has delta functions of the appropriate magnitude appearing at each jump discontinuity. This fact reassures us that the rather mysterious construction of delta functions and their generalizations is indeed the right way to extend calculus to functions which do not possess derivatives in the ordinary sense.

Integration of Fourier Series

Integration is a smoothing operation — the integrated function is always nicer than the original. Therefore, we should anticipate being able to integrate Fourier series without difficulty. There is, however, one complication: the integral of a periodic function is not necessarily periodic. The simplest example is the constant function 1, which is certainly periodic, but its integral, namely x, is not. On the other hand, integrals of all the other periodic sine and cosine functions appearing in the Fourier series are periodic. Thus, only the constant term might cause us difficulty when we try to integrate a Fourier series (12.27). According to (2.4), the constant term

$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \tag{12.67}$$

is the *mean* or *average* of the function $f(x)$ on the interval $[-\pi,\pi]$. A function has no constant term in its Fourier series if and only if it has mean zero. It is easily shown, cf. Exercise ■, that the mean zero functions are precisely the ones that remain periodic upon integration.

Lemma 12.16. *If $f(x)$ is 2π periodic, then its integral $g(x) = \int_0^x f(y)\,dy$ is 2π periodic if and only if $\int_{-\pi}^{\pi} f(x)\,dx = 0$, so that f has mean zero on the interval $[-\pi,\pi]$.*

In particular, Lemma 12.11 implies that all odd functions automatically have mean zero, and hence periodic integrals.

Since

$$\int \cos kx\,dx = \frac{\sin kx}{k}, \qquad \int \sin kx\,dx = -\frac{\cos kx}{k}, \tag{12.68}$$

termwise integration of a Fourier series without constant term is straightforward. The resulting Fourier series is given precisely as follows.

Theorem 12.17. *If f is piecewise continuous, 2π periodic, and has mean zero, then its Fourier series*

$$f(x) \sim \sum_{k=1}^{\infty} \left[\, a_k \cos kx + b_k \sin kx \,\right],$$

can be integrated term by term, to produce the Fourier series

$$g(x) = \int_0^x f(y)\,dy \sim m + \sum_{k=1}^{\infty} \left[-\frac{b_k}{k}\cos kx + \frac{a_k}{k}\sin kx \right], \tag{12.69}$$

for its periodic integral. The constant term

$$m = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,dx \tag{12.70}$$

is the mean of the integrated function.

In many situations, the integration formula (12.69) provides a very convenient alternative to the direct derivation of the Fourier coefficients.

Example 12.18. The function $f(x) = x$ is odd, and so has mean zero: $\displaystyle\int_{-\pi}^{\pi} x\,dx = 0$. Let us integrate its Fourier series

$$x \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin k\,x \tag{12.71}$$

that we found in Example 12.2. The result is the Fourier series

$$\begin{aligned} \frac{1}{2}\,x^2 &\sim \frac{\pi^2}{6} - 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos k\,x \\ &= \frac{\pi^2}{6} - 2\left(\cos x - \frac{\cos 2\,x}{4} + \frac{\cos 3\,x}{9} - \frac{\cos 4\,x}{16} + \cdots \right), \end{aligned} \tag{12.72}$$

whose the constant term is the mean of the left hand side:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{2}\,dx = \frac{\pi^2}{6}\,.$$

Let us revisit the derivation of the integrated Fourier series from a slightly different standpoint. If we were to integrate each trigonometric summand in a Fourier series (12.27) from 0 to x, we would obtain

$$\int_0^x \cos k\,y\,dy = \frac{\sin k\,x}{k}\,, \qquad \text{whereas} \qquad \int_0^x \sin k\,y\,dy = \frac{1}{k} - \frac{\cos k\,x}{k}\,.$$

The extra $1/k$ terms arising from the definite sine integrals do not appear explicitly in our previous form for the integrated Fourier series, (12.69), and so must be hidden in the constant term m. We deduce that the mean value of the integrated function can be computed using the Fourier sine coefficients of f via the formula

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(x)\,dx = m = \sum_{k=1}^{\infty} \frac{b_k}{k}. \tag{12.73}$$

For example, the result of integrating both sides of the Fourier series (12.71) for $f(x) = x$ from 0 to x is

$$\frac{x^2}{2} \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2}\,(1 - \cos k\,x).$$

The constant terms sum up to yield the mean value of the integrated function:

$$2\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ \ldots \right) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{x^2}{2}\,dx = \frac{\pi^2}{6}\,, \tag{12.74}$$

which reproduces a formula established in Exercise ■.

More generally, if $f(x)$ does not have mean zero, its Fourier series has a nonzero constant term,

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[\, a_k \cos k\,x + b_k \sin k\,x \,\right].$$

In this case, the result of integration will be

$$g(x) = \int_0^x f(y)\,dy \sim \frac{a_0}{2}\,x + m + \sum_{k=1}^{\infty} \left[-\frac{b_k}{k} \cos k\,x + \frac{a_k}{k} \sin k\,x \right], \tag{12.75}$$

where m is given in (12.73). The right hand side is not, strictly speaking, a Fourier series. There are two ways to interpret this formula within the Fourier framework. Either we can write (12.75) as the Fourier series for the difference

$$g(x) - \frac{a_0}{2}\,x \sim m + \sum_{k=1}^{\infty} \left[-\frac{b_k}{k} \cos k\,x + \frac{a_k}{k} \sin k\,x \right], \tag{12.76}$$

which is a 2π periodic function, cf. Exercise ■. Alternatively, one can replace x by its Fourier series (12.30), and the result will be the Fourier series for the 2π periodic extension of the integral $g(x) = \displaystyle\int_0^x f(y)\,dy$.

Differentiation of Fourier Series

Differentiation has the opposite effect to integration. Differentiation makes a function worse. Therefore, to justify taking the derivative of a Fourier series, we need to know that the differentiated function remains reasonably nice. Since we need the derivative $f'(x)$ to be piecewise C^1 for the convergence Theorem 12.7 to be applicable, we must require that $f(x)$ itself be continuous and piecewise C^2.

Theorem 12.19. *If f is 2π periodic, continuous, and piecewise C^2, then its Fourier series can be differentiated term by term, to produce the Fourier series for its derivative*

$$f'(x) \sim \sum_{k=1}^{\infty} \left[\, k\,b_k \cos k\,x - k\,a_k \sin k\,x \,\right]. \tag{12.77}$$

Example 12.20. The derivative (11.53) of the absolute value function $f(x) = |\,x\,|$ is the sign function

$$\frac{d}{dx}\,|\,x\,| = \operatorname{sign} x = \begin{cases} +1, & x > 0 \\ -1, & x < 0. \end{cases}$$

Therefore, if we differentiate its Fourier series (12.47), we obtain the Fourier series

$$\operatorname{sign} x \sim \frac{4}{\pi}\left(\sin x + \frac{\sin 3\,x}{3} + \frac{\sin 5\,x}{5} + \frac{\sin 7\,x}{7} + \cdots \right). \tag{12.78}$$

Note that $\operatorname{sign} x = \sigma(x) - \sigma(-x)$ is the difference of two step functions. Indeed, subtracting the step function Fourier series (12.41) at x from the same series at $-x$ reproduces (12.78).

Example 12.21. If we differentiate the Fourier series

$$x \sim 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin k\,x = 2\left(\sin x - \frac{\sin 2\,x}{2} + \frac{\sin 3\,x}{3} - \frac{\sin 4\,x}{4} + \cdots \right),$$

we obtain an apparent contradiction:

$$1 \sim 2 \sum_{k=1}^{\infty} (-1)^{k+1} \cos k\,x = 2\cos x - 2\cos 2\,x + 2\cos 3\,x - 2\cos 4\,x + \cdots . \tag{12.79}$$

But the Fourier series for 1 just consists of a single constant term! (Why?)

The resolution of this paradox is not difficult. The Fourier series (12.30) does *not* converge to x, but rather to its periodic extension $\widetilde{f}(x)$, which has a jump discontinuity of magnitude 2π at odd multiples of π; see Figure 12.1. Thus, Theorem 12.19 is *not* directly applicable. Nevertheless, we can assign a consistent interpretation to the differentiated series. The derivative $\widetilde{f}\,'(x)$ of the periodic extension is *not* equal to the constant function 1, but, rather, has an additional delta function concentrated at each jump discontinuity:

$$\widetilde{f}\,'(x) = 1 - 2\pi \sum_{j=-\infty}^{\infty} \delta\bigl(x - (2j+1)\pi \bigr) = 1 - 2\pi\, \widetilde{\delta}(x-\pi),$$

where $\widetilde{\delta}$ denotes the 2π periodic extension of the delta function, cf. (12.66). The differentiated Fourier series (12.79) does, in fact, converge to this modified distributional derivative! Indeed, differentiation and integration of Fourier series is entirely compatible with the calculus of generalized functions, as will be borne out in yet another example.

Example 12.22. Let us differentiate the Fourier series (12.41) for the step function and see if we end up with the Fourier series (12.60) for the delta function. We find

$$\frac{d}{dx}\,\sigma(x) \sim \frac{2}{\pi}\bigl(\cos x + \cos 3\,x + \cos 5\,x + \cos 7\,x + \cdots \bigr), \tag{12.80}$$

which does *not* agree with (12.60) — half the terms are missing! The explanation is similar to the preceding example: the 2π periodic extension of the step function has two jump discontinuities, of magnitudes $+1$ at even multiples of π and -1 at odd multiples. Therefore, its derivative is the difference of the 2π periodic extension of the delta function at 0, with Fourier series (12.60) minus the 2π periodic extension of the delta function at π, with Fourier series

$$\delta(x-\pi) \sim \frac{1}{2\pi} + \frac{1}{\pi}\bigl(-\cos x + \cos 2\,x - \cos 3\,x + \cdots \bigr)$$

derived in Exercise ■. The difference of these two delta function series produces (12.80).

12.4. Change of Scale.

So far, we have only dealt with Fourier series on the standard interval of length 2π. (We chose $[-\pi,\pi]$ for convenience, but all of the results and formulas are easily adapted to any other interval of the same length, e.g., $[0, 2\pi]$.) Since physical objects like bars and strings do not all come in this particular length, we need to understand how to adapt the formulas to more general intervals. The basic idea is to rescale the variable so as to stretch or contract the standard interval[†].

Any symmetric interval $[-\ell, \ell]$ of length 2ℓ can be rescaled to the standard interval $[-\pi, \pi]$ by using the linear change of variables

$$x = \frac{\ell}{\pi} y, \qquad \text{so that} \qquad -\pi \le y \le \pi \qquad \text{whenever} \qquad -\ell \le x \le \ell. \tag{12.81}$$

Given a function $f(x)$ defined on $[-\ell, \ell]$, the *rescaled function* $F(y) = f\left(\frac{\ell}{\pi} y\right)$ lives on $[-\pi, \pi]$. Let

$$F(y) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos k y + b_k \sin k y \right],$$

be the standard Fourier series for $F(y)$, so that

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \cos k y \, dy, \qquad\qquad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \sin k y \, dy. \tag{12.82}$$

Then, reverting to the unscaled variable x, we deduce that

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos \frac{k\pi x}{\ell} + b_k \sin \frac{k\pi x}{\ell} \right]. \tag{12.83}$$

The Fourier coefficients a_k, b_k can be computed directly from $f(x)$. Indeed, replacing the integration variable in (12.82) by $y = \pi x/\ell$, and noting that $dy = (\pi/\ell)\, dx$, we deduce the adapted formulae

$$a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{k\pi x}{\ell}\, dx, \qquad\qquad b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{k\pi x}{\ell}\, dx, \tag{12.84}$$

for the Fourier coefficients of $f(x)$ on the interval $[-\ell, \ell]$.

All of the convergence results, integration and differentiation formulae, etc., that are valid for the interval $[-\pi, \pi]$ carry over, essentially unchanged, to Fourier series on nonstandard intervals. In particular, adapting our basic convergence Theorem 12.7, we conclude that if $f(x)$ is piecewise C^1, then its rescaled Fourier series (12.83) converges to its 2ℓ periodic extension $\widetilde{f}(x)$, subject to the proviso that $\widetilde{f}(x)$ takes on the midpoint values at all jump discontinuities.

Example 12.23. Let us compute the Fourier series for the function $f(x) = x$ on the interval $-1 \le x \le 1$. Since f is odd, only the sine coefficients will be nonzero. We have

$$b_k = \int_{-1}^{1} x \sin k\pi x \, dx = \left[-\frac{x \cos k\pi x}{k\pi} + \frac{\sin k\pi x}{(k\pi)^2} \right]_{x=-1}^{1} = \frac{2(-1)^{k+1}}{k\pi}.$$

† The same device was already used, in Section 5.4, to adapt the orthogonal Legendre polynomials to other intervals.

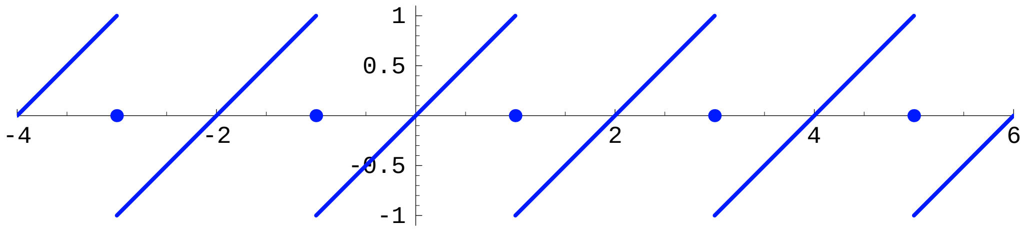

Figure 12.10. 2 Periodic Extension of x.

The resulting Fourier series is

$$x \sim \frac{2}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \cdots\right).$$

The series converges to the 2 periodic extension of the function x, namely

$$\widetilde{f}(x) = \begin{cases} x - 2m, & 2m-1 < x < 2m+1, \\ 0, & x = m, \end{cases} \qquad \text{where } m \text{ is an arbitrary integer,}$$

plotted in Figure 12.10.

We can similarly reformulate complex Fourier series on the nonstandard interval $[-\ell, \ell]$. Using (12.81) to rescale the variables in (12.55), we find

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k\, e^{\,\mathrm{i}\, k\pi x/\ell}, \qquad \text{where} \qquad c_k = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\, e^{-\,\mathrm{i}\, k\pi x/\ell}\, dx. \tag{12.85}$$

Again, this is merely an alternative way of writing the real Fourier series (12.83).

When dealing with a more general interval $[a, b]$, there are two options. The first is to take a function $f(x)$ defined for $a \le x \le b$ and periodically extend it to a function $\widetilde{f}(x)$ that agrees with $f(x)$ on $[a, b]$ and has period $b - a$. One can then compute the Fourier series (12.83) for its periodic extension $\widetilde{f}(x)$ on the symmetric interval $[\frac{1}{2}(a-b), \frac{1}{2}(b-a)]$ of width $2\ell = b - a$; the resulting Fourier series will (under the appropriate hypotheses) converge to $\widetilde{f}(x)$ and hence agree with $f(x)$ on the original interval. An alternative approach is to translate the interval by an amount $\frac{1}{2}(a+b)$ so as to make it symmetric; this is accomplished by the change of variables $\widehat{x} = x - \frac{1}{2}(a+b)$. an additional rescaling will convert the interval into $[-\pi, \pi]$. The two methods are essentially equivalent, and full details are left to the reader.

12.5. Convergence of the Fourier Series.

The purpose of this final section is to establish some basic convergence results for Fourier series. This is not a purely theoretical exercise, since convergence considerations impinge directly upon a variety of applications of Fourier series. One particularly important consequence is the connection between smoothness of a function and the decay rate of its high order Fourier coefficients — a result that is exploited in signal and image denoising and in the analytical properties of solutions to partial differential equations.

Be forewarned: the material in this section is more mathematical than we are used to, and the more applied reader may consider omitting it on a first reading. However, a full understanding of the scope of Fourier analysis as well as its limitations does requires some familiarity with the underlying theory. Moreover, the required techniques and proofs serve as an excellent introduction to some of the most important tools of modern mathematical analysis. Any effort expended to assimilate this material will be more than amply rewarded in your later career.

Unlike power series, which converge to analytic functions on the interval of convergence, and diverge elsewhere (the only tricky point being whether or not the series converges at the endpoints), the convergence of a Fourier series is a much more subtle matter, and still not understood in complete generality. A large part of the difficulty stems from the intricacies of convergence in infinite-dimensional function spaces. Let us therefore begin with a brief discussion of the fundamental issues.

Convergence in Vector Spaces

We assume that you are familiar with the usual calculus definition of the limit of a sequence of real numbers: $\lim_{n \to \infty} a_n = a^\star$. In any finite-dimensional vector space, e.g., $\mathbb{R}^m$, there is essentially only one way for a sequence of vectors $\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots \in \mathbb{R}^m$ to converge, which is guaranteed by any one of the following equivalent criteria:

(a) The vectors converge: $\mathbf{v}^{(n)} \longrightarrow \mathbf{v}^\star \in \mathbb{R}^m$ as $n \to \infty$.

(b) The individual components of $\mathbf{v}^{(n)} = (v_1^{(n)}, \ldots, v_m^{(n)})$ converge, so $\lim_{n \to \infty} v_i^{(n)} = v_i^\star$ for all $i = 1, \ldots, m$.

(c) The difference in norms goes to zero: $\| \mathbf{v}^{(n)} - \mathbf{v}^\star \| \longrightarrow 0$ as $n \to \infty$.

The last requirement, known as *convergence in norm*, does not, in fact, depend on which norm is chosen. Indeed, Theorem 3.17 implies that, on a finite-dimensional vector space, all norms are essentially equivalent, and if one norm goes to zero, so does any other norm.

The analogous convergence criteria are certainly *not the same* in infinite-dimensional vector spaces. There are, in fact, a bewildering variety of convergence mechanisms in function space, that include pointwise convergence, uniform convergence, convergence in norm, weak convergence, and many others. All play a significant role in advanced mathematical analysis, and hence all are deserving of study. Here, though, we shall be content to learn just the most basic aspects of convergence of the Fourier series, leaving further details to more advanced texts, e.g., [**63**, **159**, **199**].

The most basic convergence mechanism for a sequence of functions $v_n(x)$ is called *pointwise convergence*, which requires that

$$\lim_{n \to \infty} v_n(x) = v_\star(x) \qquad \text{for all} \quad x. \tag{12.86}$$

In other words, the functions' values at each individual point converge in the usual sense. Pointwise convergence is the function space version of the convergence of the components of a vector. Indeed, pointwise convergence immediately implies component-wise convergence of the sample vectors $\mathbf{v}^{(n)} = (\, v_n(x_1), \ldots, v_n(x_m)\,)^T \in \mathbb{R}^m$ for any choice of sample points $x_1, \ldots, x_m$.

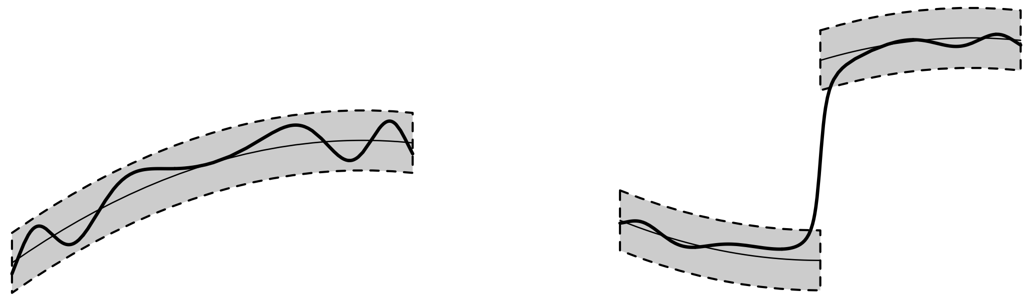

Figure 12.11. Uniform and Non-Uniform Convergence of Functions.

On the other hand, *convergence in norm* of the function sequence requires

$$\lim_{n \to \infty} \| v_n - v_\star \| = 0,$$

where $\| \cdot \|$ is a prescribed norm on the function space. As we have learned, not all norms on an infinite-dimensional function space are equivalent: a function might be small in one norm, but large in another. As a result, convergence in norm *will* depend upon the choice of norm. Moreover, convergence in norm does not necessarily imply pointwise convergence or vice versa. A variety of examples can be found in the exercises.

Uniform Convergence

Proving uniform convergence of a Fourier series is reasonably straightforward, and so we will begin there. You no doubt first saw the concept of a uniformly convergent sequence of functions in your calculus course, although chances are it didn't leave much of an impression. In Fourier analysis, uniform convergence begins to play an increasingly important role, and is worth studying in earnest. For the record, let us restate the basic definition.

Definition 12.24. A sequence of functions $v_n(x)$ is said to converge *uniformly* to a function $v_\star(x)$ on a subset $I \subset \mathbb{R}$ if, for every $\varepsilon > 0$, there exists an integer $N = N(\varepsilon)$ such that

$$|\, v_n(x) - v_\star(x) \,| < \varepsilon \qquad \text{for all} \;\; x \in I \;\; \text{and all} \;\; n \geq N. \tag{12.87}$$

The key point — and the reason for the term "uniform convergence" — is that the integer N depends only upon ε and not on the point $x \in I$. Roughly speaking, the sequence converges uniformly if and only if for any small ε, the graphs of the functions eventually lie inside a band of width 2ε centered around the graph of the limiting function; see Figure 12.11. Functions may converge pointwise, but non-uniformly: the Gibbs phenomenon is the prototypical example of a nonuniformly convergent sequence: For a given $\varepsilon > 0$, the closer x is to the discontinuity, the larger n must be chosen so that the inequality in (12.87) holds, and hence there is *no* consistent choice of N that makes (12.87) valid for all x and all $n \geq N$. A detailed discussion of these issues, including the proofs of the basic theorems, can be found in any basic real analysis text, e.g., [**9**, **158**, **159**].

A key consequence of uniform convergence is that it preserves continuity.

Theorem 12.25. *If $v_n(x) \to v_\star(x)$ converges uniformly, and each $v_n(x)$ is continuous, then $v_\star(x)$ is also a continuous function.*

The proof is by contradiction. Intuitively, if $v_\star(x)$ were to have a discontinuity, then, as sketched in Figure 12.11, a sufficiently small band around its graph would not connect together, and this prevents the graph of any continuous function, such as $v_n(x)$, from remaining entirely within the band. Rigorous details can be found in [**9**].

Warning: A sequence of continuous functions can converge *non-uniformly* to a continuous function. An example is the sequence $v_n(x) = \dfrac{2nx}{1+n^2x^2}$, which converges pointwise to $v_\star(x) \equiv 0$ (why?) but not uniformly since $\max |\, v_n(x)\,| = v_n\bigl(\frac{1}{n}\bigr) = 1$, which implies that (12.87) cannot hold when $\varepsilon < 1$.

The convergence (pointwise, uniform, in norm, etc.) of a series $\displaystyle\sum_{k=1}^{\infty} u_k(x)$ is, by definition, governed by the convergence of its sequence of *partial sums*

$$v_n(x) = \sum_{k=1}^{n} u_k(x). \tag{12.88}$$

The most useful test for uniform convergence of series of functions is known as the *Weierstrass M–test*, in honor of the nineteenth century German mathematician Karl Weierstrass, known as the "father of modern analysis".

Theorem 12.26. *Let $I \subset \mathbb{R}$. Suppose the functions $u_k(x)$ are bounded by*

$$|\, u_k(x)\,| \leq m_k \qquad \text{for all} \qquad x \in I, \tag{12.89}$$

where the $m_k \geq 0$ are fixed positive constants. If the series

$$\sum_{k=1}^{\infty} m_k < \infty \tag{12.90}$$

converges, then the series

$$\sum_{k=1}^{\infty} u_k(x) = f(x) \tag{12.91}$$

converges uniformly and absolutely[†] *to a function $f(x)$ for all $x \in I$. In particular, if the summands $u_k(x)$ in Theorem 12.26 are continuous, so is the sum $f(x)$.*

[†] Recall that a series $\displaystyle\sum_{n=1}^{\infty} a_n = a^\star$ is said to converge *absolutely* if and only if $\displaystyle\sum_{n=1}^{\infty} |\, a_n\,|$ converges, [**9**].

With some care, we are allowed to manipulate uniformly convergent series just like finite sums. Thus, if (12.91) is a uniformly convergent series, so is the term-wise product

$$\sum_{k=1}^{\infty} g(x)\, u_k(x) = g(x)\, f(x) \tag{12.92}$$

with any bounded function: $|\, g(x) \,| \leq C$ for $x \in I$. We can integrate a uniformly convergent series term by term[‡], and the resulting integrated series

$$\int_a^x \left(\sum_{k=1}^{\infty} u_k(y) \right) dy = \sum_{k=1}^{\infty} \int_a^x u_k(y)\, dy = \int_a^x f(y)\, dy \tag{12.93}$$

is uniformly convergent. Differentiation is also allowed — but only when the differentiated series converges uniformly.

Proposition 12.27. *If* $\displaystyle\sum_{k=1}^{\infty} u_k'(x) = g(x)$ *is a uniformly convergent series, then* $\displaystyle\sum_{k=1}^{\infty} u_k(x) = f(x)$ *is also uniformly convergent, and, moreover,* $f'(x) = g(x)$.

We are particularly interested in applying these results to Fourier series, which, for convenience, we take in complex form

$$f(x) \;\sim \sum_{k=-\infty}^{\infty} c_k\, e^{\,\mathrm{i}\, k\, x}. \tag{12.94}$$

Since x is real, $\left| \, e^{\,\mathrm{i}\, k\, x} \,\right| \leq 1$, and hence the individual summands are bounded by

$$\left| \, c_k\, e^{\,\mathrm{i}\, k\, x} \,\right| \leq |\, c_k \,| \qquad \text{for all } x.$$

Applying the Weierstrass M–test, we immediately deduce the basic result on uniform convergence of Fourier series.

Theorem 12.28. *If the Fourier coefficients* c_k *satisfy*

$$\sum_{k=-\infty}^{\infty} |\, c_k \,| < \infty, \tag{12.95}$$

then the Fourier series (12.94) *converges uniformly to a continuous function* $\widetilde{f}(x)$ *having the same Fourier coefficients:* $c_k = \langle\, f\, ;e^{\,\mathrm{i}\, k\, x} \,\rangle = \langle\, \widetilde{f}\, ;e^{\,\mathrm{i}\, k\, x} \,\rangle$.

Proof: Uniform convergence and continuity of the limiting function follow from Theorem 12.26. To show that the c_k actually are the Fourier coefficients of the sum, we multiply the Fourier series by $e^{-\,\mathrm{i}\, k\, x}$ and integrate term by term from $-\pi$ to π. As in (12.92, 93), both operations are valid thanks to the uniform convergence of the series. *Q.E.D.*

[‡] Assuming that the individual functions are all integrable.

The one thing that the theorem does not guarantee is that the original function $f(x)$ used to compute the Fourier coefficients c_k is the *same* as the function $\widetilde{f}(x)$ obtained by summing the resulting Fourier series! Indeed, this may very well not be the case. As we know, the function that the series converges to is necessarily 2π periodic. Thus, at the very least, $\widetilde{f}(x)$ will be the 2π periodic extension of $f(x)$. But even this may not suffice.

Two functions $f(x)$ and $\widehat{f}(x)$ that have the same values except for a finite set of points $x_1, \ldots, x_m$ have the same Fourier coefficients. (Why?) More generally, two functions which agree everywhere outside a set of "measure zero" will have the same Fourier coefficients. In this way, a convergent Fourier series singles out a distinguished representative from a collection of essentially equivalent 2π periodic functions.

Remark: The term "measure" refers to a rigorous generalization of the notion of the length of an interval to more general subsets $S \subset \mathbb{R}$. In particular, S has *measure zero* if it can be covered by a collection of intervals of arbitrarily small total length. For example, any collection of finitely many points, or even countably many points, e.g., the rational numbers, has measure zero. The proper development of the notion of measure, and the consequential Lebesgue theory of integration, is properly studied in a course in real analysis, [**158**, **159**].

As a consequence of Theorem 12.28, Fourier series cannot converge uniformly when discontinuities are present. Non-uniform convergence is typically manifested by some form of Gibbs pehnomenon at the discontinuities. However, it can be proved, [**33**, **63**, **199**], that even when the function fails to be everywhere continuous, its Fourier series is uniformly convergent on any closed subset of continuity.

Theorem 12.29. *Let $f(x)$ be 2π periodic and piecewise C^1. If f is continuous for $a < x < b$, then its Fourier series converges uniformly to $f(x)$ on any closed subinterval $a + \delta \leq x \leq b - \delta$, with $\delta > 0$.*

For example, the Fourier series (12.41) for the step function does converge uniformly if we stay away from the discontinuities; for instance, by restriction to a subinterval of the form $[\,\delta, \pi - \delta\,]$ or $[\,-\pi + \delta, -\delta\,]$ for any $0 < \delta < \frac{1}{2}\pi$. This reconfirms our observation that the nonuniform Gibbs behavior becomes progressively more and more localized at the discontinuities.

Smoothness and Decay

The uniform convergence criterion (12.95) requires, at the very least, that the Fourier coefficients decay to zero: $c_k \to 0$ as $k \to \pm\infty$. In fact, the Fourier coefficients cannot tend to zero too slowly. For example, the individual summands of the infinite series

$$\sum_{k=-\infty}^{\infty} \frac{1}{|\,k\,|^{\alpha}} \tag{12.96}$$

go to 0 as $k \to \infty$ whenever $\alpha > 0$, but the series only converges when $\alpha > 1$. (This follows from the standard integral convergence test for series, [**9**, **159**].) Thus, if we can bound the Fourier coefficients by

$$|\,c_k\,| \;\le\; \frac{M}{|\,k\,|^\alpha} \qquad \text{for all} \qquad |\,k\,| \gg 0, \tag{12.97}$$

for some power $\alpha > 1$ and some positive constant $M > 0$, then the Weierstrass M test will guarantee that the Fourier series converges uniformly to a continuous function.

An important consequence of the differentiation formulae (12.77) for Fourier series is the fact that the faster the Fourier coefficients of a function tend to zero as $k \to \infty$, the smoother the function is. Thus, one can detect the degree of smoothness of a function by seeing how rapidly its Fourier coefficients decay to zero. More rigorously:

Theorem 12.30. *If the Fourier coefficients satisfy*

$$\sum_{k=-\infty}^{\infty} k^n\,|\,c_k\,| \;<\; \infty, \tag{12.98}$$

then the Fourier series (12.55) *converges to an n times continuously differentiable 2π periodic function $f(x) \in \mathrm{C}^n$. Moreover, for any $m \le n$, the m times differentiated Fourier series converges uniformly to the corresponding derivative $f^{(m)}(x)$.*

Proof: This is an immediate consequence of Proposition 12.27 combined with Theorem 12.28. Application of the Weierstrass M test to the differentiated Fourier series based on our hypothesis (12.98) serves to complete the proof. *Q.E.D.*

Corollary 12.31. *If the Fourier coefficients satisfy* (12.97) *for some $\alpha > n+1$, then the function $f(x)$ is n times continuously differentiable.*

Thus, stated roughly, the smaller its high frequency Fourier coefficients, the smoother the function. If the Fourier coefficients go to zero faster than any power of k, e.g., exponentially fast, then the function is infinitely differentiable. Analyticity is a little more delicate, and we refer the reader to [**63**, **199**] for details.

Example 12.32. The 2π periodic extension of the function $|\,x\,|$ is continuous with piecewise continuous first derivative. Its Fourier coefficients (12.46) satisfy the estimate (12.97) for $\alpha = 2$, which is not quite fast enough to ensure a continuous second derivative. On the other hand, the Fourier coefficients (12.29) of the step function $\sigma(x)$ only tend to zero as $1/k$, so $\alpha = 1$, reflecting the fact that its periodic extension is only piecewise continuous. Finally, the Fourier coefficients (12.59) for the delta function do not tend to zero at all, indicative of the fact that it is not an ordinary function, and its Fourier series does not converge in the standard sense.

Hilbert Space

In order to make further progress, we must take a little detour. The proper setting for the rigorous theory of Fourier series turns out to be the most important function space in modern physics and modern analysis, known as *Hilbert space* in honor of the great German mathematician David Hilbert. The precise definition of this infinite-dimensional inner product space is rather technical, but a rough version goes as follows:

Definition 12.33. A complex-valued function $f(x)$ is called *square-integrable* on the interval $[-\pi,\pi]$ if it satisfies

$$\| f \|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x) |^2 \, dx < \infty. \tag{12.99}$$

The *Hilbert space* $L^2 = L^2[-\pi,\pi]$ is the vector space consisting of all complex-valued *square-integrable* functions.

Note that (12.99) is the L^2 norm based on the standard Hermitian inner product

$$\langle f ; g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, \overline{g(x)} \, dx. \tag{12.100}$$

The triangle inequality

$$\| c f + d g \| \leq | c | \, \| f \| + | d | \, \| g \|,$$

implies that the Hilbert space is, as claimed, a complex vector space, i.e., if $f, g \in L^2$, so $\| f \|, \| g \| < \infty$, then any linear combination $c f + d g \in L^2$ since $\| c f + d g \| < \infty$. The Cauchy–Schwarz inequality

$$| \langle f ; g \rangle | \leq \| f \| \, \| g \|,$$

implies that the inner product of two square-integrable functions is well-defined and finite. In particular, the Fourier coefficients of a function $f(x)$ are defined as inner products

$$c_k = \langle f ; e^{\,i k x} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-\,i k x} \, dx$$

of f with the complex exponentials (which are continuous and so in L^2), and hence are well-defined for any $f \in L^2$.

There are some interesting analytical subtleties that arise when one tries to prescribe precisely which functions are to be admitted to Hilbert space. Every piecewise continuous function belongs to L^2. But some functions with singularities are also members. For example, the power function $| x |^{-\alpha}$ belongs to L^2 for any $\alpha < \frac{1}{2}$, but not if $\alpha \geq \frac{1}{2}$.

Analysis requires limiting procedures, and Hilbert space must be "complete" in the sense that appropriately convergent[†] sequences of functions have a limit. The completeness requirement is not elementary, and relies on the development of the more sophisticated Lebesgue theory of integration, which was formalized in the early part of the twentieth century by the French mathematician Henri Lebesgue. Any function which is square-integrable in the Lebesgue sense is admitted into L^2. This includes such non-piecewise continuous functions as $\sin \dfrac{1}{x}$ and $x^{-1/3}$, as well as the strange function

$$r(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases} \tag{12.101}$$

[†] The precise technical requirement is that every *Cauchy sequence* of functions $v_k(x) \in L^2$ converges to a function $v_\star(x) \in L^2$; see Exercise ■ for details.

One soon discovers that square-integrable functions can be quite bizarre.

A second complication is that (12.99) does not, strictly speaking, define a norm once we allow discontinuous functions into the fold. For example, the piecewise continuous function

$$f_0(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0, \end{cases} \tag{12.102}$$

has norm zero, $\| f_0 \| = 0$, even though it is not zero everywhere. Indeed, any function which is zero except on a set of measure zero also has norm zero, including the function (12.101). Therefore, in order to make (12.99) into a legitimate norm on Hilbert space, we must agree to identify any two functions which have the same values except on a set of measure zero. For instance, the zero function 0 and the preceding examples $f_0(x)$ and $r(x)$ are all viewed as defining the *same* element of Hilbert space. Thus, although we treat them as if they were ordinary functions, each element of Hilbert space is not, in fact, a function, but, rather, an equivalence class of functions all differing on a set of measure zero. All this might strike the applied reader as becoming much too abstract and arcane. In practice, you will not lose much by assuming that the "functions" in L^2 are always piecewise continuous and square-integrable. Nevertheless, the full analytical power of Hilbert space theory is only unleashed by including completely general functions in L^2.

After its invention by pure mathematicians around the turn of the twentieth century, physicists in the 1920's suddenly realized that Hilbert space was the correct setting to establish the modern theory of quantum mechanics. A quantum mechanical *wave function* is a element† $\varphi \in \mathrm{L}^2$ that has unit norm: $\| \varphi \| = 1$. Thus, the set of wave functions is merely the unit sphere in Hilbert space. Quantum mechanics endows each physical wave function with a probabilistic interpretation. Suppose the wave function represents a single subatomic particle — photon, electron, etc. The modulus $| \varphi(x) |$ of the wave function quantifies the probability of finding the particle at the position x. More correctly, the probability that the particle resides in a prescribed interval $[a, b]$ is equal to $\sqrt{\dfrac{1}{2\pi} \displaystyle\int_a^b | \varphi(x) |^2 \, dx}$. In particular, the wave function has unit norm

$$\| \varphi \| = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} | \varphi(x) |^2 \, dx} = 1$$

because the particle must certainly, i.e., with probability 1, be *somewhere*!

Convergence in Norm

We are now in a position to discuss convergence in norm of the Fourier series. We begin with the basic definition, which makes sense on any normed vector space.

Definition 12.34. Let V be a normed vector space. A sequence $\mathbf{v}^{(n)}$ is said to *converge in norm* to $\mathbf{v}^\star \in V$ if $\| \mathbf{v}^{(n)} - \mathbf{v}^\star \| \to 0$ as $n \to \infty$.

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† Here we are acting as if the physical space were represented by the one-dimensional interval $[-\pi, \pi]$. The more apt case of three-dimensional physical space is developed analogously, replacing the single integral by a triple integral over all of $\mathbb{R}^3$.

As we noted earlier, on finite-dimensional vector spaces, convergence in norm is equivalent to ordinary convergence. On the other hand, on infinite-dimensional function spaces, convergence in norm is very different from pointwise convergence. For instance, it is possible, cf. Exercise ■, to construct a sequence of functions that converges in norm to 0, but does not converge pointwise *anywhere*!

We are particularly interested in the convergence in norm of the Fourier series of a square integrable function $f(x) \in \mathrm{L}^2$. Let

$$s_n(x) = \sum_{k=-n}^{n} c_k\, e^{\,\mathrm{i}\,k\,x} \tag{12.103}$$

be the n^{th} partial sum of its Fourier series (12.55). The partial sum (12.103) belongs to the subspace $\mathcal{T}^{(n)} \subset \mathrm{L}^2$ of all trigonometric polynomials of degree at most n, spanned by $e^{-\,\mathrm{i}\,n x}, \ldots, e^{\,\mathrm{i}\,n x}$. It is, in fact, distinguished as the function in $\mathcal{T}^{(n)}$ that lies the closest to f, where the distance between functions is measured by the L^2 norm of their difference: $\| f - g \|$. This important characterization of the Fourier partial sums is, in fact, an immediate consequence of the orthonormality of the trigonometric basis.

Theorem 12.35. *The n^{th} order Fourier partial sum $s_n \in \mathcal{T}^{(n)}$ is the best least squares approximation to $f \in \mathrm{L}^2$, meaning that it minimizes the distance, as measured by the L^2 norm of the difference*

$$\| f - p_n \|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\, f(x) - p_n(x) \,|^2 \, dx, \tag{12.104}$$

among all possible degree n trigonometric polynomials

$$p_n(x) = \sum_{k=-n}^{n} d_k\, e^{\,\mathrm{i}\,k\,x} \in \mathcal{T}^{(n)}. \tag{12.105}$$

Proof: The proof is, in fact, an exact replica of that of the finite-dimensional Theorems 5.37 and 5.39. Note first that, owing to the orthonormality of the basis exponentials, (12.54), we can compute the norm of a trigonometric polynomial (12.105) by summing the squared moduli of its Fourier coefficients:

$$\| p_n \|^2 = \langle\, p_n \,;p_n \,\rangle = \sum_{k,l=-n}^{n} d_k\, \overline{d_l}\, \langle\, e^{\,\mathrm{i}\,k\,x} \,; e^{\,\mathrm{i}\,l\,x} \,\rangle = \sum_{k=-n}^{n} |\, d_k \,|^2,$$

reproducing our standard formula (5.5) for the norm with respect to an orthonormal basis in this situation. Therefore, employing the identity in Exercise 3.6.43(a),

$$\begin{aligned} \| f - p_n \|^2 &= \| f \|^2 - 2\ \mathrm{Re}\, \langle\, f \,; p_n \,\rangle + \| p_n \|^2 = \| f \|^2 - 2\ \mathrm{Re} \sum_{k=-n}^{n} \overline{d_k}\, \langle\, f \,; e^{\,\mathrm{i}\,k\,x} \,\rangle + \| p_n \|^2 \\ &= \| f \|^2 - 2 \sum_{k=-n}^{n} \mathrm{Re}\, \big(\, c_k\, \overline{d_k}\,\big) + \sum_{k=-n}^{n} |\, d_k \,|^2 = \| f \|^2 - \sum_{k=-n}^{n} |\, c_k \,|^2 + \sum_{k=-n}^{n} |\, d_k - c_k \,|^2; \end{aligned}$$

the last equality results from adding and subtracting the squared norm

$$\| s_n \|^2 = \sum_{k=-n}^{n} | c_k |^2 \tag{12.106}$$

of the Fourier partial sum. We conclude that

$$\| f - p_n \|^2 = \| f \|^2 - \| s_n \|^2 + \sum_{k=-n}^{n} | d_k - c_k |^2. \tag{12.107}$$

The first and second terms on the right hand side of (12.107) are uniquely determined by $f(x)$ and hence cannot be altered by the choice of trigonometric polynomial $p_n(x)$, which only affects the final summation. Since the latter is a sum of nonnegative quantities, it is minimized by setting all the summands to zero, i.e., setting $d_k = c_k$. We conclude that $\| f - p_n \|$ is minimized if and only if $d_k = c_k$ are the Fourier coefficients, and hence the least squares minimizer is the Fourier partial sum: $p_n(x) = s_n(x)$. *Q.E.D.*

Setting $p_n = s_n$, so $d_k = c_k$, in (12.107), we conclude that the least squares error for the Fourier partial sum is

$$0 \leq \| f - s_n \|^2 = \| f \|^2 - \| s_n \|^2 = \| f \|^2 - \sum_{k=-n}^{n} | c_k |^2.$$

Therefore, the Fourier coefficients of the function f must satisfy the basic inequality

$$\sum_{k=-n}^{n} | c_k |^2 \leq \| f \|^2.$$

Consider what happens in the limit as $n \to \infty$. Since we are summing a sequence of non-negative numbers with uniformly bounded partial sums, the limiting summation must exist, and be subject to the same bound. We have thus proved *Bessel's inequality*:

$$\sum_{k=-\infty}^{\infty} | c_k |^2 \leq \| f \|^2, \tag{12.108}$$

which is an important waystation on the road to the general theory. Now, as noted earlier, if a series is to converge, the individual summands must go to zero: $| c_k |^2 \to 0$. Therefore, Bessel's inequality immediately implies the following simplified form of the *Riemann–Lebesgue Lemma*.

Lemma 12.36. *If $f \in \mathrm{L}^2$ is square integrable, then its Fourier coefficients satisfy*

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-\mathrm{i}\,k\,x}\, dx \quad \longrightarrow \quad 0 \qquad \text{as} \qquad | k | \to \infty, \tag{12.109}$$

which is equivalent to the decay of the real Fourier coefficients

$$\left.\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos k\,x\,dx \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin k\,x\,dx \end{aligned}\right\} \quad\longrightarrow\quad 0 \qquad \textit{as} \qquad k\to\infty. \tag{12.110}$$

Remark: As before, the convergence of the sum (12.108) requires that the coefficients c_k cannot tend to zero too slowly. For instance, assuming the power bound (12.97), namely

$$|\,c_k\,| \le M\,|\,k\,|^{-\alpha}, \text{ then requiring } \alpha > \tfrac{1}{2} \text{ is enough to ensure that } \sum_{k=-\infty}^{\infty} |\,c_k\,|^2 < \infty.$$

Thus, as we should expect, convergence in norm imposes less restrictive requirements on the decay of the Fourier coefficients than uniform convergence — which needed $\alpha > 1$. Indeed, a Fourier series may very well converge in norm to a discontinuous function, which is not possible under uniform convergence. In fact, there even exist bizarre continuous functions whose Fourier series do not converge uniformly, even failing to converge at all at some points. A deep result says that the Fourier series of a continuous function converges except possibly on a set of measure zero, [**199**]. Again, the subtle details of the convergence of Fourier series are rather delicate, and lack of space and analytical savvy prevents us from delving any further into these topics.

Completeness

As we know, specification of a basis enables you to prescribe all elements of a finite-dimensional vector space as linear combinations of the basis elements. The number of basis elements dictates the dimension. In an infinite-dimensional vector space, there are, by definition, infinitely many linearly independent elements, and no finite collection can serve to describe the entire space. The question then arises to what extent an infinite collection of linearly independent elements can be considered as a basis for the vector space. Mere counting will no longer suffice, since omitting one, or two, or any finite number — or even certain infinite subcollections — from a purported basis will still leave infinitely many linearly independent elements; but, clearly, the reduced collection should, in some sense, no longer serve as a complete basis. The curse of infinity strikes again! For example, while the complete trigonometric collection $1, \cos x, \sin x, \cos 2\,x, \sin 2\,x, \ldots$ will represent any 2π periodic L^2 function as a Fourier series, the subcollection $\cos x, \sin x, \cos 2\,x, \sin 2\,x, \ldots$ can only represent functions with mean zero, while the subcollection $\sin x, \sin 2\,x, \ldots$ only represents odd functions. All three consist of infinitely many linearly independent functions, but only the first could possibly be deemed a basis of L^2. In general, just because we have found a infinite collection of independent elements in an infinite-dimensional vector space, how do we know that we have enough, and are not missing one or two or 10,000 or even infinitely many additional independent elements?

The concept of "completeness" serves to properly formalize the notion of a "basis" of an infinite-dimensional vector space. We shall discuss completeness in a general, abstract setting, but the key example is, of course, the Hilbert space L^2 and the system of trigonometric (or complex exponential) functions forming a Fourier series. Other important examples arising in later applications include wavelets, Bessel functions, Legendre polynomials, spherical harmonics, and other systems of eigenfunctions of self-adjoint boundary value problems.

For simplicity, we only define completeness in the case of orthonormal systems. (Similar arguments will clearly apply to orthogonal systems, but normality helps to streamline the presentation.) Let V be an infinite-dimensional complex[†] inner product space. Suppose that $u_1, u_2, u_3, \ldots \in V$ form an orthonormal collection of elements of V, so

$$\langle u_i \,;u_j \rangle = \begin{cases} 1 & i = j, \\ 0, & i \neq j. \end{cases} \tag{12.111}$$

A straightforward argument, cf. Proposition 5.4, proves that the u_i are linearly independent. Given $f \in V$, we form its *generalized Fourier series*

$$f \;\sim\; \sum_{k=1}^{\infty} c_k\, u_k, \qquad \text{where} \qquad c_k = \langle f \,; u_k \rangle, \tag{12.112}$$

which is our usual orthonormal basis coefficient formula (5.4), and is obtained by formally taking the inner product of the series with u_k and invoking the orthonormality conditions (12.111).

Definition 12.37. An orthonormal system $u_1, u_2, u_3, \ldots \in V$ is called *complete* if the generalized Fourier series (12.112) of any $f \in V$ converges in norm to f:

$$\| f - s_n \| \;\longrightarrow\; 0, \qquad \text{as } n \to \infty, \qquad \text{where} \qquad s_n = \sum_{k=1}^{n} c_k\, u_k \tag{12.113}$$

is the n^{th} partial sum of the generalized Fourier series (12.112).

Thus, completeness requires that every element can be arbitrarily closely approximated (in norm) by a suitable linear combination of the basis elements. A complete orthonormal system should be viewed as the infinite-dimensional version of an orthonormal basis of a finite-dimensional vector space.

The key result for classical Fourier series is that the complex exponentials, or, equivalently, the trigonometric functions, form a complete system. An indication of its proof will appear below.

Theorem 12.38. *The complex exponentials $e^{\,i\,k\,x}$, $k = 0, \pm 1, \pm 2, \ldots$, form a complete orthonormal system in $L^2 = L^2[-\pi, \pi]$. In other words, if $s_n(x)$ denotes the n^{th} partial sum of the Fourier series of the square-integrable function $f(x) \in L^2$, then $\lim_{n \to \infty} \| f - s_n \| = 0$.*

[†] The results are equally valid in real inner product spaces, with slightly simpler proofs.

In order to understand completeness, let us describe some equivalent characterizations. The *Plancherel formula* is the infinite-dimensional counterpart of our formula (5.5) for the norm of a vector in terms of its coordinates with respect to an orthonormal basis.

Theorem 12.39. *The orthonormal system $u_1, u_2, u_3, \ldots \in V$ is complete if and only if the Plancherel formula*

$$\| f \|^2 = \sum_{k=1}^{\infty} | c_k |^2 = \sum_{k=1}^{\infty} \langle f ; u_k \rangle^2, \tag{12.114}$$

holds for every $f \in V$.

Proof: We begin by computing[†] the Hermitian norm

$$\| f - s_n \|^2 = \| f \|^2 - 2 \operatorname{Re} \langle f ; s_n \rangle + \| s_n \|^2.$$

Substituting the formula (12.113) for the partial sums, we find, by orthonormality,

$$\| s_n \|^2 = \sum_{k=1}^{n} | c_k |^2, \qquad \text{while} \qquad \langle f ; s_n \rangle = \sum_{k=1}^{n} \overline{c_k} \langle f ; u_k \rangle = \sum_{k=1}^{n} | c_k |^2.$$

Therefore,

$$0 \leq \| f - s_n \|^2 = \| f \|^2 - \sum_{k=1}^{n} | c_k |^2. \tag{12.115}$$

The fact that the left hand side of (12.115) is non-negative for all n implies the abstract form of *Bessel inequality*

$$\sum_{k=1}^{\infty} | c_k |^2 \ \leq \ \| f \|^2, \tag{12.116}$$

which is valid for *any* orthonormal system of elements in an inner product space. The trigonometric Bessel inequality (12.108) is a particular case of this general result. As we noted above, Bessel's inequality implies that the generalized Fourier coefficients $c_k \to 0$ must tend to zero reasonably rapidly in order that the sum of their squares converges.

Plancherel's Theorem 12.39, thus, states that the system of functions is complete if and only if the Bessel inequality is, in fact, an equality! Indeed, letting $n \to \infty$ in (12.115), we have

$$\lim_{n \to \infty} \| f - s_n \|^2 \ = \ \| f \|^2 - \sum_{k=1}^{\infty} | c_k |^2.$$

Therefore, the completeness condition (12.113) holds if and only if the right hand side vanishes, which is the Plancherel identity (12.114). *Q.E.D.*

[†] We are, in essence, repeating the proofs of Theorem 12.35 and the subsequent trigonometric Bessel inequality (12.108) in a more abstract setting.

Corollary 12.40. *Let $f, g \in V$. Then their Fourier coefficients* $c_k = \langle f ; \varphi_k \rangle, d_k = \langle g ; \varphi_k \rangle$ *satisfy* Parseval's identity

$$\langle f ; g \rangle = \sum_{k=1}^{\infty} c_k \overline{d_k}. \tag{12.117}$$

Proof: Using the identity in Exercise 3.6.43(*b*),

$$\langle f ; g \rangle = \tfrac{1}{4} \left(\| f + g \|^2 - \| f - g \|^2 + \mathrm{i} \, \| f + \mathrm{i} \, g \|^2 - \mathrm{i} \, \| f - \mathrm{i} \, g \|^2 \right).$$

Parseval's identity results from applying the Plancherel formula (12.114) to each term on the right hand side:

$$\langle f ; g \rangle = \frac{1}{4} \sum_{k=-\infty}^{\infty} \left(| c_k + d_k |^2 - | c_k - d_k |^2 + \mathrm{i} \, | c_k + \mathrm{i} \, d_k |^2 - \mathrm{i} \, | c_k - \mathrm{i} \, d_k |^2 \right) = \sum_{k=-\infty}^{\infty} c_k \overline{d_k},$$

by a straightforward algebraic manipulation. *Q.E.D.*

In particular, in the case of the complex exponential basis of $\mathrm{L}^2[-\pi, \pi]$, the Plancherel and Parseval formulae tell us that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x) |^2 \, dx = \sum_{k=-\infty}^{\infty} | c_k |^2, \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, \overline{g(x)} \, dx = \sum_{k=-\infty}^{\infty} c_k \overline{d_k}, \tag{12.118}$$

in which $c_k = \langle f ; e^{\mathrm{i} k k} \rangle$, $d_k = \langle g ; e^{\mathrm{i} k k} \rangle$ are the ordinary Fourier coefficients of the complex-valued functions $f(x)$ and $g(x)$. Note that the Plancherel formula is a special case of the Parseval identity, obtained by setting $f = g$. In Exercise ■, you are asked to rewrite these formulas in terms of the real Fourier coefficients.

Completeness also tells us that a function is uniquely determined by its Fourier coefficients.

Proposition 12.41. *If the orthonormal system $u_1, u_2, \ldots \in V$ is complete, then the only element $f \in V$ with all zero Fourier coefficients, $0 = c_1 = c_2 = \cdots$, is the zero element*: $f = 0$. *More generally, two elements $f, g \in V$ have the same Fourier coefficients if and only if they are the same*: $f = g$.

Proof: The proof is an immediate consequence of the Plancherel formula. Indeed, if $c_k = 0$, then (12.114) implies that $\| f \| = 0$. The second statement follows by applying the first to their difference $f - g$. *Q.E.D.*

Another way of stating this result is that the only function which is orthogonal to every element of a complete orthonormal system is the zero function†. Interpreted in yet another way, a complete orthonormal system is maximal in the sense that no further orthonormal elements can be appended to it.

† Or, to be more technically accurate, any function which is zero outside a set of measure zero.

Let us now discuss the completeness of the Fourier trigonometric/complex exponential functions. We shall prove the completeness criterion only for continuous functions, leaving the harder general proof to the references, [**63**, **199**]. According to Theorem 12.28, if $f(x)$ is continuous, 2π periodic, and piecewise C^1, its Fourier series converges uniformly to $f(x)$, so

$$f(x) \;=\; \sum_{k=-\infty}^{\infty} c_k\, e^{\,i\,k\,x} \qquad \text{for all} \qquad -\pi \le x \le \pi.$$

The same holds for its complex conjugate $\overline{f(x)}$. Therefore,

$$|\, f(x)\,|^2 = f(x)\,\overline{f(x)} = f(x) \sum_{k=-\infty}^{\infty} \overline{c}_k\, e^{-\,i\,k\,x} = \sum_{k=-\infty}^{\infty} \overline{c}_k\, f(x)\, e^{-\,i\,k\,x},$$

which also converges uniformly by (12.92). Equation (12.93) permits us to integrate both sides from $-\pi$ to π, yielding

$$\|\, f\,\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\, f(x)\,|^2\, dx = \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{c}_k\, f(x)\, e^{-\,i\,k\,x}\, dx = \sum_{k=-\infty}^{\infty} c_k\,\overline{c}_k = \sum_{k=-\infty}^{\infty} |\, c_k\,|^2.$$

Therefore, Plancherel's identity (12.114) holds for any continuous function. With some additional technical work, this result is used to establish the validity of Plancherel's formula for all $f \in L^2$, the key step being to suitably approximate f by continuous functions. With this in hand, completeness is an immediate consequence of Theorem 12.39. *Q.E.D.*

Pointwise Convergence

Let us finally turn to the proof of the Pointwise Convergence Theorem 12.7. The goal is to prove that, under the appropriate hypotheses on $f(x)$, namely 2π periodic and piecewise C^1, the limit of the partial Fourier sums is

$$\lim_{n\to\infty} s_n(x) = \tfrac{1}{2}\left[\, f(x^+) + f(x^-)\,\right]. \tag{12.119}$$

We begin by substituting the formulae (12.56) for the complex Fourier coefficients into the formula (12.103) for the n^{th} partial sum:

$$\begin{aligned} s_n(x) = \sum_{k=-n}^{n} c_k\, e^{\,i\,k\,x} &= \sum_{k=-n}^{n} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, e^{-\,i\,k\,y}\, dy \right) e^{\,i\,k\,x} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \sum_{k=-n}^{n} e^{\,i\,k(x-y)}\, dy. \end{aligned}$$

The n^{th} partial sum

$$s_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{\,i\,k\,x} = \frac{1}{2\pi}\left(e^{-\,i\,n\,x} + \;\cdots\; + e^{-\,i\,x} + 1 + e^{\,i\,x} + \;\cdots\; + e^{\,i\,n\,x} \right)$$

can, in fact, be explicitly evaluated. Recall the formula for the sum of a geometric series

$$\sum_{k=0}^{m} a\, r^k = a + a\, r + a\, r^2 + \;\cdots\; + a\, r^m = a\, \left(\frac{r^{m+1} - 1}{r - 1} \right). \tag{12.120}$$

The partial sum $s_n(x)$ has this form, with $m+1 = 2n+1$ summands, initial term $a = e^{-\,\mathrm{i}\, n x}$, and ratio $r = e^{\,\mathrm{i}\, x}$. Therefore,

$$\begin{aligned} s_n(x) = \frac{1}{2\pi} \sum_{k=-n}^{n} e^{\,\mathrm{i}\, k x} &= \frac{1}{2\pi}\, e^{-\,\mathrm{i}\, n x} \left(\frac{e^{\,\mathrm{i}\,(2n+1)x} - 1}{e^{\,\mathrm{i}\, x} - 1} \right) = \frac{1}{2\pi}\, \frac{e^{\,\mathrm{i}\,(n+1)x} - e^{-\,\mathrm{i}\, n x}}{e^{\,\mathrm{i}\, x} - 1} \\ &= \frac{1}{2\pi}\, \frac{e^{\,\mathrm{i}\,\left(n+\frac{1}{2}\right)x} - e^{-\,\mathrm{i}\,\left(n+\frac{1}{2}\right)x}}{e^{\,\mathrm{i}\, x/2} - e^{-\,\mathrm{i}\, x/2}} = \frac{1}{2\pi}\, \frac{\sin\left(n + \frac{1}{2}\right) x}{\sin \frac{1}{2} x}\,. \end{aligned} \tag{12.121}$$

In this computation, to pass from the first to the second line, we multiplied numerator and denominator by $e^{-\,\mathrm{i}\, x/2}$, after which we used the equation for the sine function in terms of complex exponentials. Incidentally, (12.121) is equivalent to the intriguing trigonometric summation formula

$$s_n(x) = \frac{1}{2\pi} + \frac{1}{\pi}\, \bigl(\cos x + \cos 2x + \cos 3x + \;\cdots\; + \cos n x \bigr) = \frac{1}{2\pi}\, \frac{\sin\left(n + \frac{1}{2}\right) x}{\sin \frac{1}{2} x}. \tag{12.122}$$

We can then use the geometric summation formula (12.121) to evaluate the result:

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, \frac{\sin\left(n + \frac{1}{2}\right)(x - y)}{\sin \frac{1}{2}(x - y)}\, dy \\ &= \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x + y)\, \frac{\sin\left(n + \frac{1}{2}\right) y}{\sin \frac{1}{2} y}\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x + y)\, \frac{\sin\left(n + \frac{1}{2}\right) y}{\sin \frac{1}{2} y}\, dy. \end{aligned}$$

The second equality is the result of changing the integration variable from y to $x + y$; the final equality follows since the integrand is 2π periodic, and so its integrals over *any* interval of length 2π all have the same value; see Exercise ■.

Thus, to prove (12.119), it suffices to show that

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{\pi} \int_{0}^{\pi} f(x + y)\, \frac{\sin\left(n + \frac{1}{2}\right) y}{\sin \frac{1}{2} y}\, dy &= f(x^+), \\ \lim_{n \to \infty} \frac{1}{\pi} \int_{-\pi}^{0} f(x + y)\, \frac{\sin\left(n + \frac{1}{2}\right) y}{\sin \frac{1}{2} y}\, dy &= f(x^-). \end{aligned} \tag{12.123}$$

The proofs of the two formulae are identical, and so we concentrate on the first. Since the integrand is even,

$$\frac{1}{\pi} \int_{0}^{\pi} \frac{\sin\left(n + \frac{1}{2}\right) y}{\sin \frac{1}{2} y}\, dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\sin\left(n + \frac{1}{2}\right) y}{\sin \frac{1}{2} y}\, dy = 1,$$

by equation (12.65). Multiplying this formula by $f(x^+)$ and subtracting off the right hand side leads to

$$\lim_{n \to \infty} \frac{1}{\pi} \int_0^\pi \frac{f(x+y) - f(x^+)}{\sin \frac{1}{2} y} \sin \left(n + \tfrac{1}{2}\right) y \; dy = 0, \tag{12.124}$$

which we now proceed to prove.

We claim that, for each fixed value of x, the function

$$g(y) = \frac{f(x+y) - f(x^+)}{\sin \frac{1}{2} y}$$

is piecewise continuous for all $0 \le y \le \pi$. Owing to our hypotheses on $f(x)$, the only problematic point is when $y = 0$, but then, by l'Hôpital's rule (for one-sided limits),

$$\lim_{y \to 0^+} g(y) = \lim_{y \to 0^+} \frac{f(x+y) - f(x^+)}{\sin \frac{1}{2} y} = \lim_{y \to 0^+} \frac{f'(x+y)}{\frac{1}{2} \cos \frac{1}{2} y} = 2 f'(x^+).$$

Consequently, (12.124) will be established if we can show that

$$\lim_{n \to \infty} \frac{1}{\pi} \int_0^\pi g(y) \sin \left(n + \tfrac{1}{2}\right) y \; dy = 0 \tag{12.125}$$

whenever g is piecewise continuous. Were it not for the extra $\frac{1}{2}$, this would immediately follow from the simplified Riemann–Lebesgue Lemma 12.36. More honestly, we can invoke the addition formula for $\sin \left(n + \frac{1}{2}\right) y$ to write

$$\frac{1}{\pi} \int_0^\pi g(y) \sin \left(n + \tfrac{1}{2}\right) y \, dy = \frac{1}{\pi} \int_0^\pi \left(g(y) \sin \tfrac{1}{2} y \right) \cos n y \, dy + \frac{1}{\pi} \int_0^\pi \left(g(y) \cos \tfrac{1}{2} y \right) \sin n y \, dy$$

The first integral is the n^{th} Fourier cosine coefficient for the piecewise continuous function $g(y) \sin \frac{1}{2} y$, while the second integral is the n^{th} Fourier sine coefficient for the piecewise continuous function $g(y) \cos \frac{1}{2} y$. Lemma 12.36 implies that both of these converge to zero as $n \to \infty$, and hence (12.125) holds. This completes the proof, establishing pointwise convergence of the Fourier series. *Q.E.D.*

Remark: An alternative approach to the last part of the proof is to use the general *Riemann–Lebesgue Lemma*, whose proof can be found in [**63**, **199**].

Lemma 12.42. *Suppose $g(x)$ is piecewise continuous on $[a, b]$. Then*

$$0 = \lim_{\omega \to \infty} \int_a^b g(x) \, e^{\,\mathrm{i}\, \omega x} \, dx = \int_a^b g(x) \cos \omega x \, dx + \mathrm{i} \int_a^b g(x) \sin \omega x \, dx. \tag{12.126}$$

Intuitively, as the frequency ω gets larger and larger, the increasingly rapid oscillations in $e^{\,\mathrm{i}\, \omega x}$ tend to cancel each other out. In mathematical language, the Riemann–Lebesgue formula (12.126) says that, as $\omega \to \infty$, the integrand $g(x) \, e^{\,\mathrm{i}\, \omega x}$ converges weakly to 0; we saw the same phenomenon in the weak convergence of the Fourier series of the delta function (12.61).

