The left inverse of a matrix $\underline{\underline{A}}$ is also the right inverse. $\underline{\underline{L}}$ is the left inverse of $\underline{\underline{A}}$

$\underline{\underline{L}}\,\underline{\underline{A}} = \underline{\underline{I}}$. $\underline{\underline{R}}$ is the right inverse : $\underline{\underline{A}}\,\underline{\underline{R}} = \underline{\underline{I}}$

$$(\underline{\underline{L}}\,\underline{\underline{A}})\underline{\underline{R}} = \underline{\underline{L}}(\underline{\underline{A}}\,\underline{\underline{R}})$$

$$\underline{\underline{I}}\,\underline{\underline{R}} = \underline{\underline{L}}\,\underline{\underline{I}} \Rightarrow \underline{\underline{R}} = \underline{\underline{L}}$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1} \quad \Big\| \quad (\underline{\underline{A}}\,\underline{\underline{B}})^{T} = \underline{\underline{B}}^{T}\underline{\underline{A}}^{T}$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1}(\underline{\underline{A}}\,\underline{\underline{B}}) \overset{?}{=} \underline{\underline{I}} \checkmark$$

$$\underline{\underline{B}}^{-1}(\underline{\underline{A}}^{-1}\underline{\underline{A}})\underline{\underline{B}} \overset{?}{=} \underline{\underline{I}}$$

$$= \underline{\underline{B}}^{-1}\underline{\underline{I}}\,\underline{\underline{B}} = \underline{\underline{B}}^{-1}\underline{\underline{B}} = \underline{\underline{I}}$$

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e.g. $\underline{\underline{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$ Find $\underline{\underline{A}}^{-1}$

$$\left(\begin{array}{ccc|ccc} \multicolumn{3}{c|}{A} & \multicolumn{3}{c}{I} \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 5 & 5 & 1 & 0 & 0 & 1 \end{array}\right) \qquad r_3 - 5r_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & -4 & -5 & 0 & 1 \end{array}\right) \qquad \frac{r_2}{2}, \; \frac{r_3}{-4}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] \qquad r_1 - r_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 3/2 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] \qquad r_1 + \frac{1}{2} r_3 \;, \quad r_2 - \frac{3}{2} r_3$$

$$\underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.}_{I_3} \left|\begin{array}{ccc} 13/8 & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & -\frac{1}{4} \end{array}\right] \longleftarrow A^{-1}$$

$\exists \underline{\underline{A}}^{-1}$ iff $\det(\underline{\underline{A}}) \neq 0$

$\underline{\underline{A}}\,\vec{x} = \vec{0}$ Linear $(n \times n)$ system homogeneous

If A^{-1} exists, then $(\underline{\underline{A}}^{-1}\underline{\underline{A}})\vec{x} = \underline{\underline{A}}^{-1}\vec{0} = \vec{0}$

$\underline{\underline{I}}\,\vec{x} = \vec{x} = \vec{0}$

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non-homogeneous $\underline{\underline{A}}\,\vec{x} = \vec{b}$

If $\exists A^{-1}$ then $\vec{x} = \underline{\underline{A}}^{-1}\vec{b} \longleftarrow$

If $\det(\underline{\underline{A}}) = 0 \Rightarrow \nexists A^{-1}$ then $\vec{x} = \vec{x}_p + \vec{x}_c$

$\underline{\underline{A}}\,\vec{x}_c = \vec{0}$

Levi-Civita totally antisymmetric symbol

e.g. two dimensions

$$\epsilon_{ij} \quad , \quad i,j \in \{1,2\} \qquad \underline{\underline{\epsilon}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

e.g. three dimensions

$$\epsilon_{ijk} \quad , \quad i,j,k \in \{1,2,3\}$$

$$\epsilon_{123} = +1 = \epsilon_{231} = \epsilon_{312}$$

$$\epsilon_{132} = -1 = \epsilon_{321} = \epsilon_{213}$$

all the rest are zero

$$\epsilon_{112} = 0 \quad , \quad \epsilon_{333} = 0 \qquad \epsilon_{111} = 0$$

Determinant

sum of the product of components, one per row and one per column, with sign given by permutation order.

e.g. ~~[illegible]~~ 2x2 matrix

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad , \quad \det(\underline{\underline{A}}) = a_{11}a_{22} - a_{12}a_{21}$$

$$\det(\underline{\underline{A}}) = |\underline{\underline{A}}| = \sum_{i=1}^{2}\sum_{j=1}^{2} \epsilon_{ij}\, \underbrace{a_{1i}\, a_{2j}}_{a_{1j}\,a_{2i}} \qquad \underbrace{}_{a_{i1}\,a_{j2}}$$

$$= \cancel{\epsilon_{11}}^{\,0}\, a_{11}a_{21} + \epsilon_{12}^{\,+1}\, a_{11}a_{22}$$

$$+ \underset{-1}{\epsilon_{21}}\, a_{12}a_{21} + \cancel{\epsilon_{22}}^{\,0}\, a_{12}a_{22}$$

three by three

$$\underline{\underline{B}} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{matrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{matrix}$$

$$\det(\underline{\underline{B}}) = |\underline{\underline{B}}| = \sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk}\, b_{1i}\, b_{2j}\, b_{3k}$$

$\underline{\underline{S}}^T = \underline{\underline{S}}$ symmetric

$\underline{\underline{A}}^T = -\underline{\underline{A}}$ antisymmetric

$\underline{\underline{O}}^T = \underline{\underline{O}}^{-1}$ orthogonal $\quad \underline{\underline{O}}^T \underline{\underline{O}} = \underline{\underline{I}}$ (identity matrix)

$\underline{\underline{R}}^* = \underline{\underline{R}}$ real

$\underline{\underline{H}}^\dagger = \underline{\underline{H}}$ Hermitian $\quad$ ($\dagger$ dagger $= T\,*$) (transpose; complex conjugate)

$\underline{\underline{U}}^\dagger = \underline{\underline{U}}^{-1}$ unitary. $\quad \underline{\underline{U}}^\dagger \underline{\underline{U}} = \underline{\underline{I}}$

$$\det(\underline{\underline{A}}^T) = \det(\underline{\underline{A}})$$

$$\det(\underline{\underline{I}}_{n\times n}) = +1 \longleftrightarrow \det(-\underline{\underline{I}}_{n\times n}) = (-1)^n$$

$$\det(\underline{\underline{A}}^{-1}) = \frac{1}{\det(\underline{\underline{A}})}$$

$$\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}}) \cdot \det(\underline{\underline{B}})$$

orthogonal matrices have $\det(\underline{\underline{O}}) = \pm 1$

$$\underline{\underline{O}}\,\underline{\underline{O}}^{-1} = \mathbb{I} \qquad \underline{\underline{O}}\,\underline{\underline{O}}^{T} = \mathbb{I}$$

$$\det(\underline{\underline{O}}\,\underline{\underline{O}}^{T}) = \det(\underline{\underline{O}})\cdot\det(O^{T}) = \det(\mathbb{I})$$

$$[\det(O)]^2 = +1$$

$$\det(O) = \pm 1 \begin{cases} +1 & \text{proper rotations} \\ -1 & \text{rotations with reflection.} \end{cases}$$

Eigenvalue Problem

In general $\underline{\underline{A}}\,\vec{x} = \vec{y}$ — $\vec{x}$ + $\vec{y}$ have different directions

$$\underset{\text{matrix}}{\underline{\underline{A}}}\;\underset{\text{vector}}{\vec{x}} = \underset{\text{scalar}}{\lambda}\;\underset{\text{vector}}{\vec{x}}$$

$\vec{x}$ = eigenvector

λ = eigen ~~vector~~ value

$$\underline{\underline{A}}\,\vec{x} = \lambda\,\underline{\underline{\mathbb{I}}}\,\vec{x} \Longrightarrow \underline{\underline{A}}\,\vec{x} - \lambda\underline{\underline{\mathbb{I}}}\,\vec{x} = \vec{0}$$

$$\Longrightarrow (\underline{\underline{A}} - \lambda\underline{\underline{\mathbb{I}}})\,\vec{x} = \vec{0}$$

If $(\underline{\underline{A}} - \lambda\underline{\underline{\mathbb{I}}})$ has an inverse $\Longrightarrow \vec{x} = \vec{0}$

non-trivial solutions exist if $\det(\underline{\underline{A}} - \lambda\underline{\underline{\mathbb{I}}}) = 0$

given $\underline{\underline{A}}$, find λ_i $\vec{x}_i$

$$\underline{\underline{A}} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find eigenvalues and eigenvectors

$$\underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta & 0 \\ \sin\theta & \cos\theta - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix}$$

$$0 = \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (-1)^{3+3}(1-\lambda) \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix}$$

$$= (1-\lambda)\left[(\cos\theta - \lambda)^2 + \sin^2\theta\right] = 0$$

$$= (1-\lambda)\left[\underbrace{\cos^2\theta + \sin^2\theta}_{1} + \lambda^2 - 2\lambda\cos\theta\right]$$

$$= (1-\lambda)\left[\lambda^2 - 2\lambda\cos\theta + 1\right] = 0$$ characteristic equation

roots: $\lambda_1 = 1$, $\lambda_2 = ?$ $\lambda_3 = ?$

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eigenvectors

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\vec{x}_1 = \vec{0}$$

$(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ has order 3 but only rank 2.

In general you only want $\hat{x}$ which has $(n-1)$ components. $\hat{x} \cdot \hat{x} = 1$, unit vector.

$$\begin{pmatrix} \cos\theta - 1 & -\sin\theta & 0 \\ \sin\theta & \cos\theta - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(\cos\theta - 1)x \quad - \sin\theta\, y \quad + 0z \quad = 0$$

$$\sin\theta\, x \quad + (\cos\theta - 1)y \quad + 0z = 0$$

$$0x \quad + 0y \quad + 0z \quad = 0$$

independent of $z \Rightarrow$ pick one say $z = 10$

$$(\cos\theta - 1)x - \sin\theta\, y = 0$$

Solution $\Rightarrow$ $x = 0$, and $y = 0$

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eigenvector $\vec{x}_1$ corresponding to $\lambda_1 = 1$

$$\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ 10 \end{pmatrix} \qquad \hat{x}_1 = \frac{\vec{x}_1}{\sqrt{\vec{x}_1 \cdot \vec{x}_1}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$