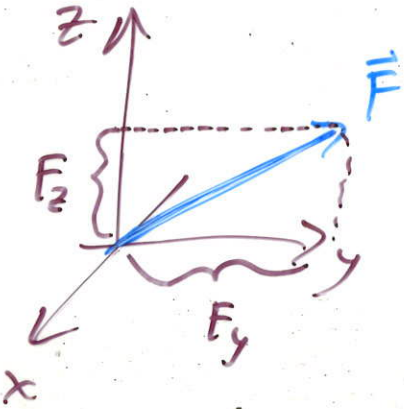
Fourier Series

Analogy with vector decomposition.



Expand $\vec{F}$ in terms of coefficients * basis vectors

Pick a set of basis vectors

e.g. $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

$= \{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$

$= \{\hat{x}, \hat{y}, \hat{z}\} = \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$

$$\vec{F} = \sum_{i=1}^{3} a_i \, \hat{e}_i = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

coefficients — basis vectors.

real number (in general, complex)

e.g. $\vec{F} = 3\hat{e}_1 + 5\hat{e}_2 + 7\hat{e}_3$

Orthonormality:

Kronecker Delta

$\hat{e}_1 \cdot \hat{e}_1 = 1$

$\hat{e}_1 \cdot \hat{e}_2 = 0$ ... (9 total)

$\hat{e}_1 \cdot \hat{e}_3 = 0$ really just 6

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

A different orthonormal basis set

$\vec{u}_1 = (1, 0, -1) = 1\hat{e}_1 + 0\hat{e}_2 + (-1)\hat{e}_3$

$\vec{u}_2 = (1, 1, 1)$

$\vec{u}_3 = (1, -2, 1)$ $\quad\quad \vec{u}_1 \cdot \vec{u}_1 = 2$

Normalize

$$\hat{u}_1 = \frac{\vec{u}_1}{\sqrt{\vec{u}_1 \cdot \vec{u}_1}} \quad \text{so} \quad \hat{u}_1 \cdot \hat{u}_1 = \frac{\vec{u}_1 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = 1$$

$$\begin{cases} \hat{u}_1 = \frac{1}{\sqrt{2}}(1, 0, -1) = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right) \\ \hat{u}_2 = \frac{1}{\sqrt{3}}(1, 1, 1) \\ \hat{u}_3 = \frac{1}{\sqrt{6}}(1, -2, 1) \end{cases}$$

$$\vec{F} = \sum_{j=1}^{3} b_j \hat{u}_j = b_1 \hat{u}_1 + b_2 \hat{u}_2 + b_3 \hat{u}_3$$

Find the new coefficients $\{b_j\}$.

Use orthonormality: $\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$

$$\hat{u}_i \cdot \vec{F} = \hat{u}_i \cdot \sum_{j=1}^{3} b_j \hat{u}_j = \sum_{j=1}^{3} b_j \, \hat{u}_i \cdot \hat{u}_j$$

$$= \sum_{j=1}^{3} b_j \, \delta_{ij} = b_i$$

$$b_1 = \hat{u}_1 \cdot \vec{F} = \frac{1}{\sqrt{2}}(1, 0, -1) \cdot (3, 5, 7) = \frac{-4}{\sqrt{2}}$$

$$b_2 = \hat{u}_2 \cdot \vec{F} = \frac{1}{\sqrt{3}}(1, 1, 1) \cdot (3, 5, 7) = \frac{+15}{\sqrt{3}}$$

$$b_3 = \hat{u}_3 \cdot \vec{F} = \frac{1}{\sqrt{6}}(1, -2, 1) \cdot (3, 5, 7) = 0$$

Check:

$$\vec{F} = \sum_{k=1}^{3} b_k \hat{u}_k$$

$$= \frac{-4}{\sqrt{2}} \frac{1}{\sqrt{2}}(1, 0, -1) + \frac{15}{\sqrt{3}} \frac{1}{\sqrt{3}}(1, 1, 1) + \underbrace{\cancel{0 \frac{1}{\sqrt{6}}(1, -2, 1)}}_{\to 0}$$

$$= (3, 5, 7)$$

Task: Find a basis set $\{\hat{v}_k\}$ such that $\vec{F} = c_1 \hat{v}_1 + 0\,\hat{v}_2 + 0\,\hat{v}_3$

$$\hat{v}_1 = \frac{1}{\sqrt{83}}(3, 5, 7)$$

$$\hat{v}_2 = \frac{1}{\sqrt{58}}(7, 0, -3)$$

$$\hat{v}_3 = \hat{v}_1 \times \hat{v}_2$$

Completeness

(dual to orthonormality)

$$\vec{F} = \sum_{p=1}^{3} b_p \hat{u}_p \qquad b_p = \hat{u}_p \cdot \vec{F} = \vec{F} \cdot \hat{u}_p$$

$$\vec{F} = \sum_{p=1}^{3} (\vec{F} \cdot \hat{u}_p)\, \hat{u}_p = \vec{F} \cdot \left[\sum_{p=1}^{3} \underbrace{\hat{u}_p \hat{u}_p}_{\text{dyad notation}} \right]$$

sometimes $\hat{u}_p \wedge \hat{u}_p$

$\hat{u}_p \hat{u}_p$ is an outer product $\rightarrow$ matrix (3x3)

$\hat{u}_p \cdot \hat{u}_p$ is an inner product $\rightarrow$ scalar (1x1)

$\hat{u}_p \times \hat{u}_k$ is the cross product $\rightarrow$ vector (3x1)

Row and Column vectors

$$\vec{F} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}, \quad \hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \hat{u}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Dot product: $\hat{u}_1 \cdot \hat{u}_1 = \frac{1}{\sqrt{2}} \underbrace{(1, 0, -1)^*}_{1 \times 3} \frac{1}{\sqrt{2}} \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{3 \times 1} = 1$ $= 1 \times 1$

Outer product: $\hat{e}_2 \wedge \hat{e}_1 = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{3 \times 1} \underbrace{(1\ 0\ 0)^*}_{1 \times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $= 3 \times 3$

~~Now with functions~~

$$\vec{F} = \vec{F} \cdot \sum_{p=1}^{3} \hat{u}_p \wedge \hat{u}_p = \vec{F} \cdot \mathbb{I}_{3\times 3}$$ identity matrix

check
$$\sum_{p=1}^{3} \hat{u}_p \wedge \hat{u}_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \hat{u}_1 \wedge \hat{u}_1^T + \hat{u}_2 \wedge \hat{u}_2^T + \hat{u}_3 \wedge \hat{u}_3^T$$ transpose (hermitian conjugate in general)

$$= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}(1, 0, -1) + \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(1, 1, 1) + \frac{1}{6}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}(1, -2, 1)$$

$$= \underline{\frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & +1 \end{pmatrix}} + \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \underline{\frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ -2 & +4 & -2 \\ 1 & -2 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \{\hat{u}_1, \hat{u}_2, \hat{u}_3\}$ spans 3-dimensional space

(all linearly independent)