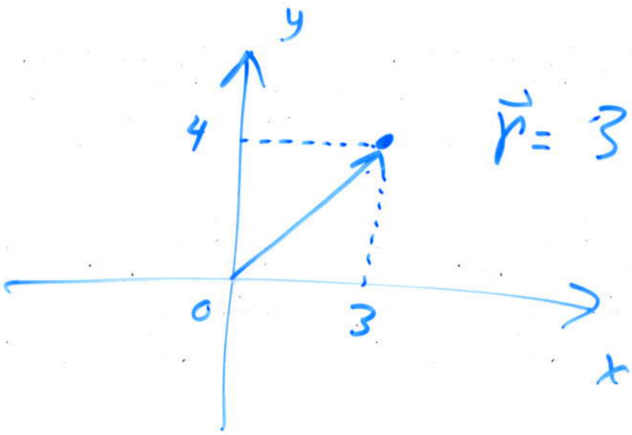
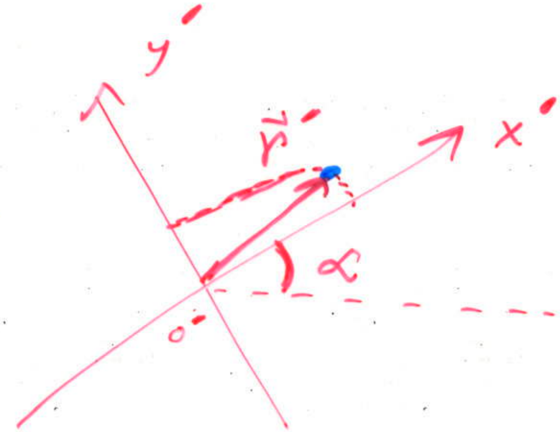
Vectors in 2 dimensions



$$\vec{r} = 3\hat{e}_x + 4\hat{e}_y$$



$$x' = \underline{\;x\;} \cos\alpha + \underline{\;y\;} \sin\alpha$$

$$y' = \underline{\;y\;} \cos\alpha + \underline{\;-x\;} \sin\alpha$$

Divergence

$$\left.\begin{matrix} V_1(\vec{r}) \\ V_2(\vec{r}) \\ V_3(\vec{r}) \end{matrix}\right\} \longrightarrow \boxed{\text{Div}} \longrightarrow S(\vec{r})$$

vector $\qquad\qquad$ scalar

$$\vec{\nabla}\cdot\vec{V}(\vec{r}) = S(\vec{r})$$

$$\text{div}\left[\vec{V}(\vec{r})\right] = S(\vec{r})$$

Cartesian

$$\vec{\nabla}\cdot\vec{V}(x,y,z) = \frac{\partial V_x(x,y,z)}{\partial x} + \frac{\partial V_y(x,y,z)}{\partial y} + \frac{\partial V_z(x,y,z)}{\partial z}$$

e.g. $\vec{V} = \hat{e}_x \sin(x) + \hat{e}_y z^2 y + \hat{e}_z e^{xy}$

$$\vec{\nabla}\cdot\vec{V} = \cos(x) + z^2 + 0 = S(\vec{r}) = \cos(x) + z^2$$

Generalized Coordinates

$V_1(q_1 q_2 q_3)$ $\quad h_2(q_1 q_2 q_3)$ $\quad h_3(q_1 q_2 q_3)$

$$\vec{\nabla}\cdot\vec{V}(q_1,q_2,q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(V_1 h_2 h_3\right) + \frac{\partial}{\partial q_2}\left(V_2 h_3 h_1\right) + \frac{\partial}{\partial q_3}\left(V_3 h_1 h_2\right)\right]$$

Spherical Polar

$$\vec{\nabla}\cdot\vec{V}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(V_r r^2\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(V_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}.$$

e.g.

$$\vec{V}(\vec{r}) = \frac{\vec{r}}{|\vec{r}|^2} \xrightarrow{\text{cartesian}} \hat{e}_x \frac{x}{x^2+y^2+z^2} + \hat{e}_y \frac{y}{x^2+y^2+z^2} + \hat{e}_z \frac{z}{x^2+y^2+z^2}$$

$$\vec{\nabla} \cdot \vec{V}(x,y,z) = ?$$

Look at one term

$$\frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2+z^2}\right] = \frac{1}{x^2+y^2+z^2} + \frac{-2x^2}{(x^2+y^2+z^2)^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

2nd term: $\frac{x^2-y^2+z^2}{(x^2+y^2+z^2)^2}$ third term $\frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$

$$\vec{\nabla} \cdot \vec{V}(x,y,z) = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2} = \frac{1}{|\vec{r}|^2}$$

$$\vec{V}(r,\theta,\varphi) = \hat{e}_r \, \frac{1}{r} + \hat{e}_\theta \, \emptyset + \hat{e}_\varphi \, \emptyset$$

$$\vec{\nabla} \cdot \vec{V}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}(V_r r^2) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r} r^2\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(r)$$

$$= \frac{1}{r^2}(1) = \frac{1}{r^2}$$

Divergence theorem (Gauss, Ostrogradsky)

$$\underset{\substack{\text{Volume}\\ \mathcal{V}}}{\iiint} \vec{\nabla}\cdot\vec{V}(\vec{r})\, d\mathcal{V} = \underset{\substack{\text{surface}\\ S=\partial\mathcal{V}}}{\oiint} \vec{V}(\vec{r})\cdot d\vec{A}$$

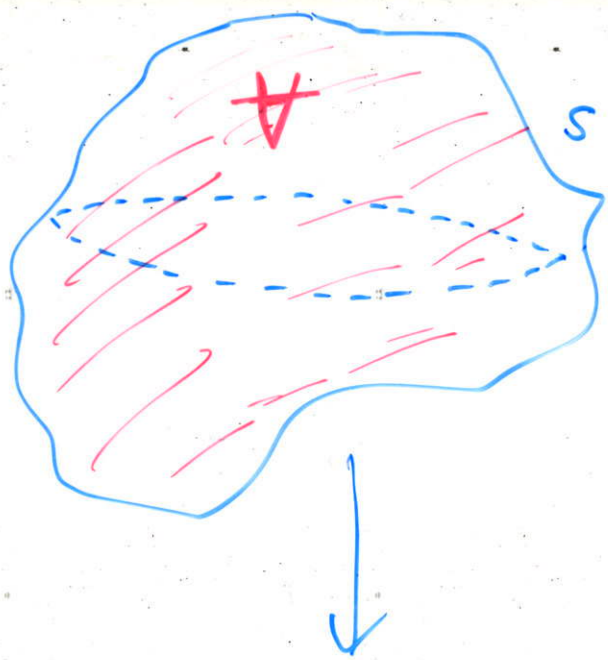
open

$dx\,dy\,dz$

$r^2\sin\theta\, dr\, d\theta\, d\varphi$

closed

$dx\,dy$, $dy\,dz$, $r^2\sin\theta\, d\theta\, d\varphi$

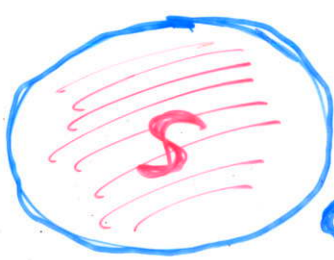


Closed surface S bounds the open volume $\mathcal{V}$

$S = \partial\mathcal{V}$

$\partial S = \partial\partial\mathcal{V} = 0$

$\partial\partial M = 0$



Closed ~~line~~ curve C bounds the open area S

$C = \partial S$

$\partial C = 0 = \partial\partial S$

$\partial C \neq 0$

Curl

$$\vec{\nabla} \times \vec{V}(\vec{r}) = \vec{W}(\vec{r})$$

$$\text{curl}\left[\vec{V}(\vec{r})\right] = \vec{W}(\vec{r})$$

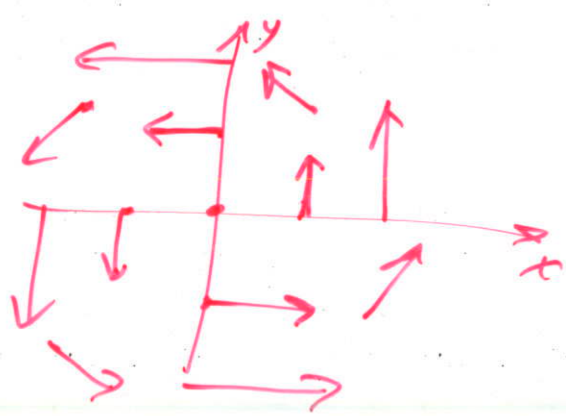
$$\text{rot}\left[\vec{V}(\vec{r})\right] = \vec{W}(\vec{r})$$

$$\vec{\nabla} \times \vec{V}(\vec{r}) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial/\partial x & \frac{\partial}{\partial y} & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix} \longleftarrow \text{means determinant}$$

$$= (-1)^{1+1} \hat{e}_x \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_y & V_z \end{vmatrix} + - \hat{e}_y \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_x & V_z \end{vmatrix} + \hat{e}_z \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_x & V_y \end{vmatrix}$$

$$= \hat{e}_x\left[\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right] - \hat{e}_y\left[\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right] + \hat{e}_z\left[\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right]$$

e.g. $\vec{V}(x,y,z) = -y\,\hat{e}_x + x\,\hat{e}_y$



$$\vec{\nabla} \times \vec{V}(\vec{r}) = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{vmatrix}$$

$$= \hat{e}_x[0] + - \hat{e}_y[0] + \hat{e}_z\left[\frac{\partial x}{\partial x} - \frac{\partial(-y)}{\partial y}\right] = 2\,\hat{e}_z$$

Generalized Coordinates

$$\vec{\nabla} \times \vec{V}(q_1 q_2 q_3) = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ V_1 h_1 & V_2 h_2 & V_3 h_3 \end{vmatrix}$$

Spherical Polar

$$h_r = 1 \qquad h_\theta = r \qquad h_\varphi = r \sin\theta$$

$$\vec{\nabla} \times \vec{V}(r,\theta,\varphi) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta r & \hat{e}_\varphi r \sin\theta \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \varphi \\ V_r & V_\theta r & V_\varphi r \sin\theta \end{vmatrix}$$

Stokes' Theorem

$$\underset{\substack{\text{surface} \\ S}}{\iint} \left[\vec{\nabla} \times \vec{V}(\vec{r})\right] \cdot d\vec{A} = \underset{\substack{\text{curve} \\ C}}{\oint} \vec{V}(\vec{r}) \cdot d\vec{l}$$

open (surface S) closed (curve C)