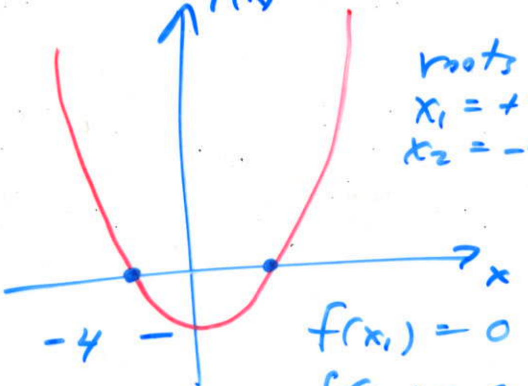
Complex Numbers

$x^2 + 1 = 0$

Why? Real numbers are not closed under algebra.

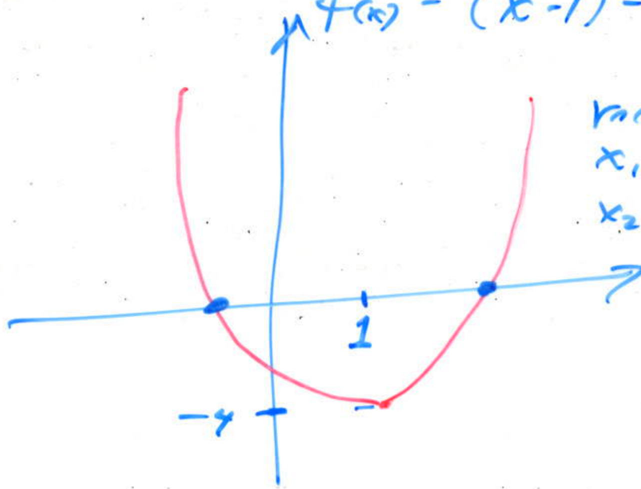
$f(x) = x^2 - 4$



roots
$x_1 = +2$
$x_2 = -2$

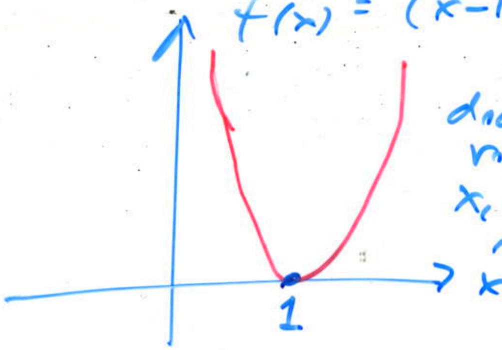
$f(x_1) = 0$
$f(x_2) = 0$

$f(x) = (x-1)^2 - 4$



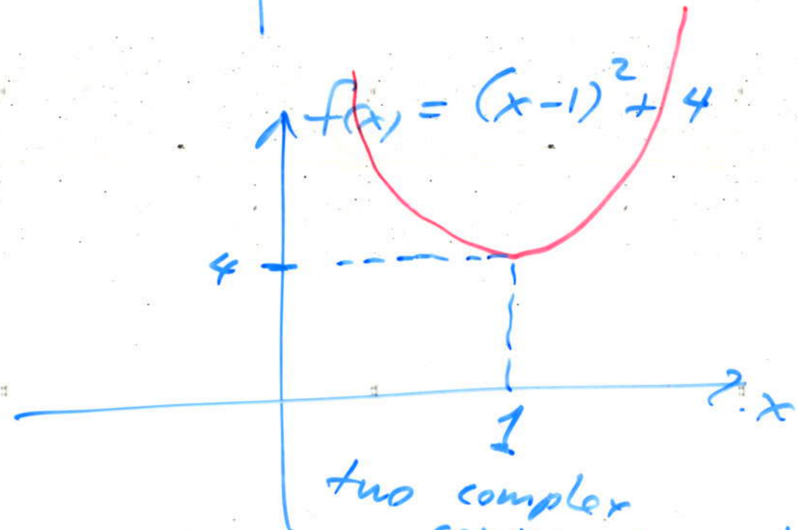
roots
$x_1 = +3$
$x_2 = -1$

$f(x) = (x-1)^2$



double root
$x_1 = 1$
$x_2 = 1$

$f(x) = (x-1)^2 + 4$



two complex conjugate roots
$x_1 = 1 + 2i$
$x_2 = 1 - 2i$

Gerolamo Cardano, 1545, Simon Stevin 1594

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubic (Cardano) $x^3 + px + q = 0$ (not x^2)

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Sometimes, get negative numbers under the square root.

e.g. $p = -15$, $q = -4$

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{-2 + \sqrt{-121}}$$

three roots all real

$x_1 = 4$

$x_2 = -2 + \sqrt{3}$

$x_3 \; -2 - \sqrt{3}$

Bombelli: $\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$

$$a + \sqrt{-1}\,b \qquad\qquad a - \sqrt{-1}\,b$$

$$2 + i \qquad\qquad 2 - i$$

Need $\sqrt{-1}$ even if all the roots are real!

Complex Variables $\quad z \in \mathbb{C}$ set of complex numbers

$$\left.\begin{aligned} z &= x + iy \quad \text{(Cartesian)} \\ &= r e^{i\theta} \quad \text{(Polar)} \end{aligned}\right\} \quad x, y, r, \theta \in \mathbb{R} \text{ set of real number}$$

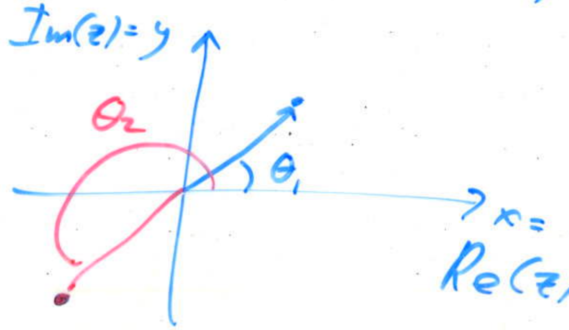
$r \geq 0$ positive semidefinite

$$i = \sqrt{-1}, \quad i^2 = -1, \quad (-i)^2 = -1.$$

$$e^{i\theta} = \cos\theta + i \sin\theta$$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$ be careful!



$$\left.\begin{aligned} \theta_1 &= \arctan\left(\tfrac{1}{1}\right) = \frac{\pi}{4} \\ \theta_2 &= \arctan\left(\tfrac{-1}{-1}\right) = \frac{\pi}{4} + \pi \end{aligned}\right\} \pm 2\pi n$$

$n \in \mathbb{Z}$

integers

Complex conjugate $i \to -i$

$$\bar{z} = z^* = x - iy = re^{-i\theta}$$

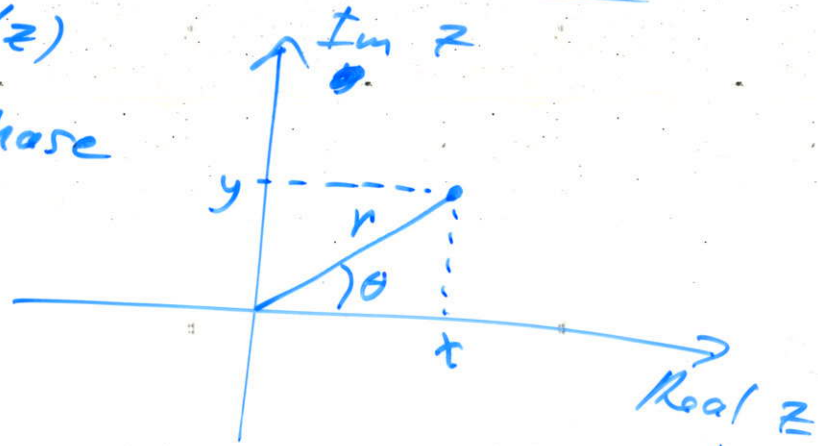
$$x = \frac{z+z^*}{2} \qquad y = \frac{z-z^*}{2i}$$

modulus of z: (magnitude) $|z| \equiv \sqrt{z^* z} = \sqrt{(x-iy)(x+iy)}$

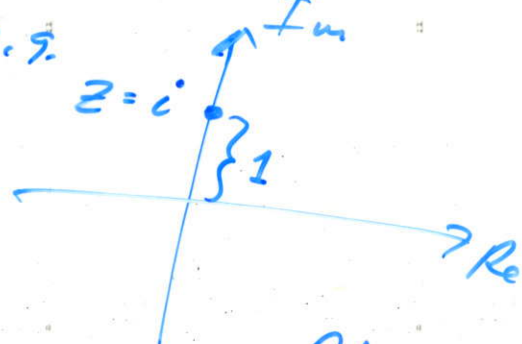
$$= \sqrt{x^2+y^2} = r = \mathrm{mod}(z)$$

$\theta = \arg(z)$ or $\mathrm{ph}(z)$

argument — phase



e.g.



$\mathrm{mod}(i) = 1$

$\arg(i) = \pi/2 \; (+2\pi n)$

Polar form: $x+iy = 0 + 1i = i$

$$re^{i\theta} = 1\, e^{\frac{\pi}{2}i + 2\pi i n}$$

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e.g.

$$-1 = e^{\pi i} \implies e^{i\pi} + 1 = 0$$

Homework: i^i

$z = re^{i\theta}$ $w = se^{i\varphi}$

$\uparrow$ $\uparrow$

affix affix

$$zw = rs\,e^{i(\theta+\varphi)}$$

$$\mathrm{mod}(zw) = \mathrm{mod}(z) * \mathrm{mod}(w)$$

$$\arg(zw) = \arg(z) + \arg(w)$$

$$(z+w)^* = z^* + w^* \qquad (zw)^* = z^* w^*$$

$z = x+iy$ $w = u+iv$

$$z+w = (x+u) + i(y+v)$$

$$zw = (x+iy)(u+iv) = (xu - yv) + i(xv + yu)$$

Inverse $\dfrac{1}{z} = \dfrac{1}{z}\dfrac{z^*}{z^*} = \dfrac{z^*}{|z|^2}$

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

4 Division Algebras

Reals

Complex Numbers

Quaternions

Octonions.

Commutativity $ab = ba$

Associativity $a(bc) = (ab)c$

$$\sin(ix) = i\sinh(x)$$

$$\cos(ix) = \cosh(x)$$

Complex Functions of Complex Variables.

$$f_{(z)} = \underset{\substack{\uparrow \\ Re(f)}}{u(z)} + i \underset{\substack{\uparrow \\ Im(f)}}{v(z)} = u(x,y) + i\, v(x,y)$$

$u, v, x, y \in \mathbb{R}$

e.g. extract the roots of unity

$$z^2 = 1 \Rightarrow (re^{i\theta})^2 = r^2 e^{i2\theta} = 1 = e^{0+2\pi n i}$$

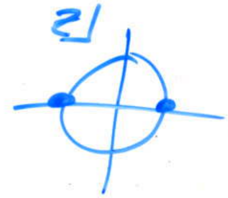
$r = 1$

$2\theta = 2\pi n \qquad n = 0, \pm 1, \pm 2, \pm 3 \ldots$

$\theta = 0, \pi \qquad n=0,\ n=1$

$r_1 = 1e^0 = +1 \quad , \quad r_2 = 1e^{\pi i} = -1$

z|



$(n=2)\ r_3 = 1e^{2\pi i} = +1 \quad , \quad r_4 = 1e^{3\pi i} = -1 \ (n=3)$

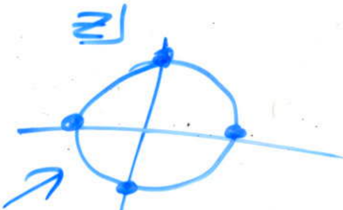
Fourth roots of unity $\quad z^4 = 1 \Rightarrow r^4 e^{i4\theta} = 1e^{0+2\pi n i}$

$n = 0$	1	2	3	$\ldots$
$\theta = 0$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\ldots$

$n=1$

$z_1 = 1 \quad z_2 = i \quad z_3 = -1 \ , \ z_4 = -i \ \ldots$

z|



unit circle $|z| = 1$

$x^2 + y^2 = 1$

Derivatives

$$f'(z)\Big|_{z=z_0} = \frac{df}{dz}\Big|_{z=z_0} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

In order for $f'(z)$ to be unique:

$$f'(z)\Big|_{z=z_0} = \lim_{x \to x_0}\left[\frac{u(x,y_0) - u(x_0,y_0)}{x-x_0} + i\frac{v(x,y_0) - v(x_0,y_0)}{x-x_0}\right]$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big|_{z=z_0}$$

$$f'(z)\Big|_{z=z_0} = \lim_{y \to y_0}\left[\frac{u(x_0,y) - u(x_0,y_0)}{i(y-y_0)} + i\frac{v(x_0,y) - v(x_0,y_0)}{i(y-y_0)}\right]$$

$\frac{1}{i} = -i$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\Big|_{z=z_0}$$

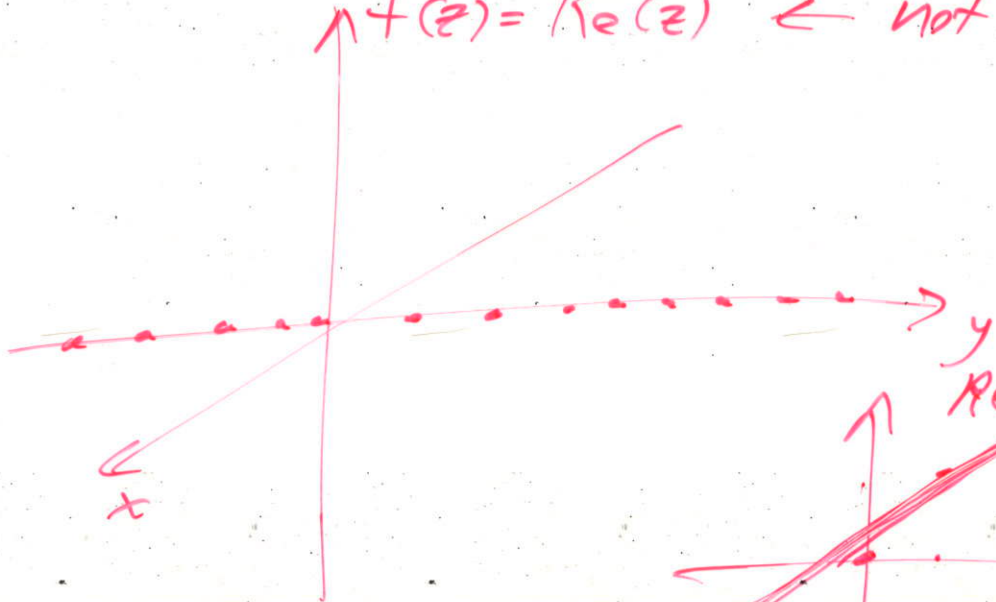
Derivative is unique if

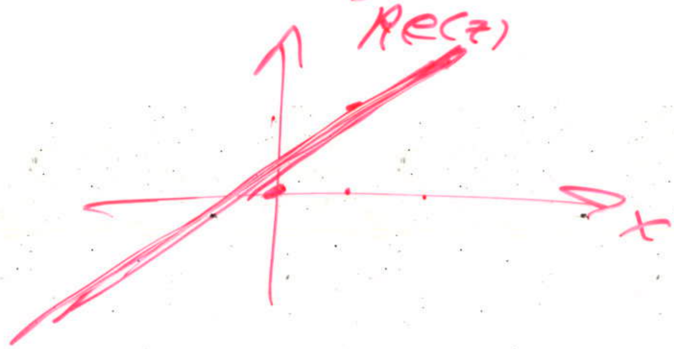
$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Longrightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Cauchy-Riemann Equations

$f(z) = Re(z)$ ← not <u>analytic</u>





Win bar ~~bets~~ bets with complex numbers

$$\left(\underbrace{18^2 + 78^2}_{(x^2+y^2)}\right)\left(\underbrace{8^2 + 24^2}_{(u^2+v^2)}\right) =$$

$$4101120 = a^2+b^2 = c^2+d^2$$

$$= (\underline{x+iy})(x-iy)(\underline{u+iv})(u-iv)$$

$$= \underbrace{\left[(x+iy)(u+iv)\right]}_{(\underbrace{a+ib}_{z})} \cdot \underbrace{\left[(x-iy)(u-iv)\right]}_{(\underbrace{a-ib}_{z^*})} = a^2+b^2$$

$$a = xu - yv = 18 \cdot 8 - 78 \cdot 24 = -1728$$

$$b = yu + xv = 78 \cdot 8 + 18 \cdot 24 = 1056$$

check: $(1728)^2 + (1056)^2 = ?$

$$\underbrace{\left[(x+iy)(u-iv)\right]}_{(c+id)} \cdot \underbrace{\left[(x-iy)(u+iv)\right]}_{(c-id)} = c^2+d^2$$

$$c = xu + yv = 18 \cdot 8 + 78 \cdot 24 = 2016$$

$$d = yu - xv = 78 \cdot 8 \cancel{+} - 18 \cdot 24 = 192$$

$$(2016)^2 + (192)^2 = ?$$

Cauchy-Riemann Equations $f = u + iv$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Is $f(z) = z^2$ analytic? Yes

$$f(z) = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + i\,\underbrace{2xy}_{v} = u(x,y) + i\,v(x,y)$$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} \checkmark \qquad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = +2y \checkmark$$

Is $f(z) = e^z$ analytic? Yes.

$$f(z) = e^{x+iy} = e^x \cdot e^{iy} = e^x\left[\cos(y) + i\sin(y)\right]$$
$$= u(x,y) + iv(x,y)$$

$$u = e^x\cos(y) \qquad v = e^x\sin(y)$$

$$\frac{\partial u}{\partial x} = e^x\cos(y) = \frac{\partial v}{\partial y} = e^x\cos(y) \checkmark$$

$$\frac{\partial u}{\partial y} = -e^x\sin(y) \qquad \frac{\partial v}{\partial x} = e^x\sin(y) \checkmark$$

If $f = |z|^2$ analytic?

$$f = zz^* = x^2 + y^2 = u(x,y) \quad , \quad v(x,y) = 0$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 0 \quad \times$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial x} = 0 \quad \times$$

If f can be written explicitly as a function of z (not z^*), then f is analytic.

Analytic – has an infinite number of derivatives and the derivative are unique.

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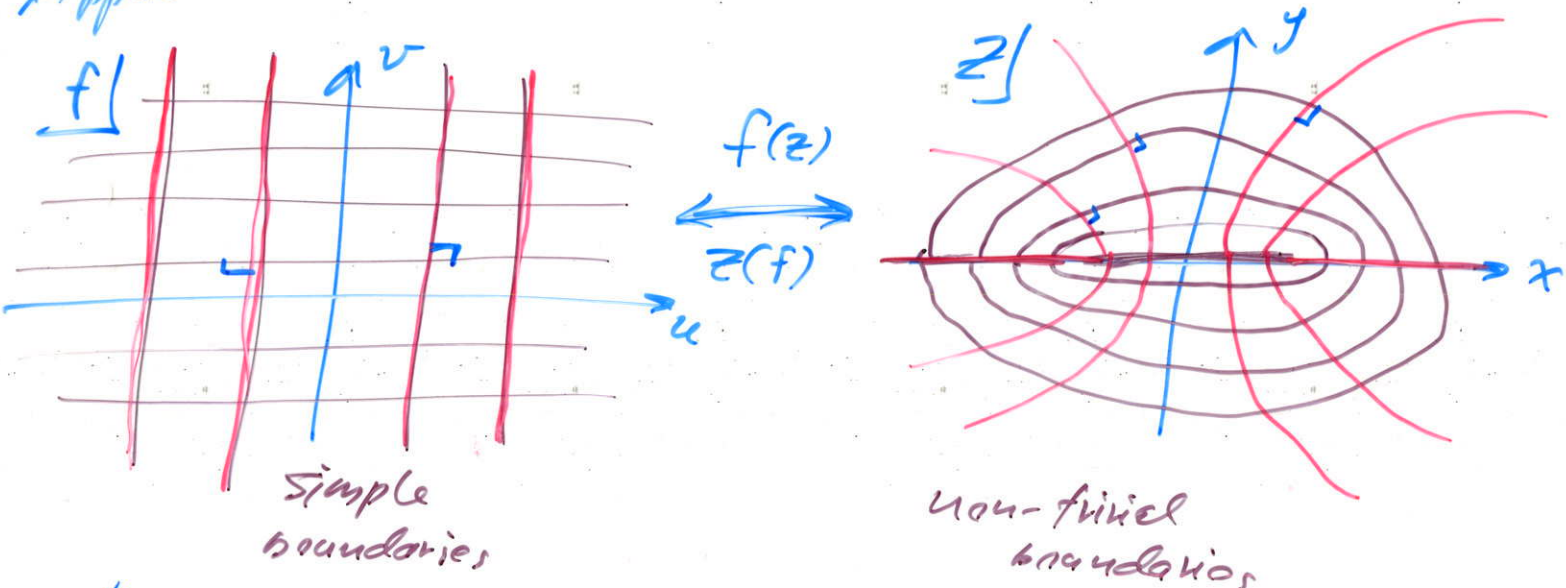
f is entire if it is analytic over the whole complex plane.

If $f(z)$ is not analytic at the point $z = z_0$
Then z_0 is a singular point.

Application: Conformal Mapping (preserves angles)



Lines of constant u are perpendicular to lines of constant v.

Cauchy – Riemann $\quad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \quad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\dfrac{\partial}{\partial x} \quad \dfrac{\partial}{\partial y} \qquad \dfrac{\partial}{\partial y} \quad \dfrac{\partial}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{add}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \boxed{0 = \nabla^2 u(x,y)}$$

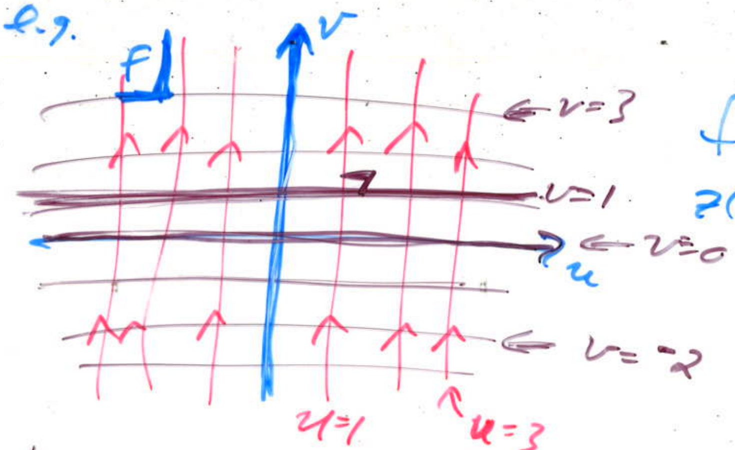
2-dimensional Laplace's differential equation.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\nabla^2 V(x,y) = 0$$

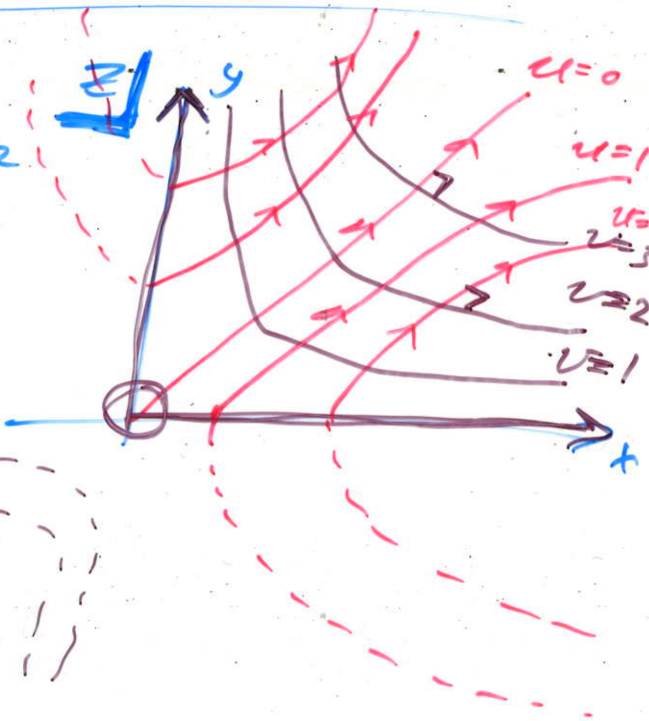
u and v are called conjugate harmonic functions.

e.g.



$f(z) = z^2$

$z(f) = \sqrt{f}$



lines of constant v (black) are equipotential lines.

lines of electric field (red) $\vec{E}$.

$$f = u + iv = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u} + i \underbrace{2xy}_{v}$$

$u = 0 \Rightarrow x^2 - y^2 = 0 \Rightarrow x^2 = y^2 \Rightarrow x = \pm y$

$v = 1 \Rightarrow 2xy = 1$ hyperbola

$v = 2 \Rightarrow 2xy = 2$

$u = 1 \Rightarrow x^2 - y^2 = 1$

$v = 0 \Rightarrow 2xy = 0 \Rightarrow x = 0$ or $y = 0$

$f = z^4 \Rightarrow \angle \ \pi/4 = 45°$

$f(z) = z^{\pi/\theta} \quad \angle \theta$