Physics 4321/7305 Professor Scalise
Old Homework Assignment Spring 2021

4321

1. (a) Using separation of variables, solve the one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} = 0$$

for the temperature u at position x and time t along a thin metal rod that sits between $x = 0$ and $x = a$. The ends of the rod are in contact with an ice water (0° Celsius) reservoir and at time zero, the middle of the rod from $x = a/4$ to $x = 3a/4$ is heated to 100° C.

   (b) Make plots of the temperature versus distance for a few times or a single three-dimensional plot of (x,t,u) for the problem above.

7305

1. (a) Solve the two-dimensional wave equation

$$\frac{\partial^2 \psi(x,y,t)}{\partial t^2} - c^2\left[\frac{\partial^2 \psi(x,y,t)}{\partial x^2} + \frac{\partial^2 \psi(x,y,t)}{\partial y^2}\right] = 0$$

for the axially symmetric oscillations of a circular drumhead with radius a, where ψ is the displacement of the drumhead from its equilibrium height.

   (b) What are the lowest three frequencies of oscillation?

   (c) Make plots of the drumheads in the first three modes of oscillation.

Bonus: Solve as much of the other class' assignment as you can.

① Assume a solution $u(x,t) = R(x)\,\theta(t)$

then the one-dimensional heat equation is

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow R(x)\frac{d\theta(t)}{dt} - k\frac{d^2R(x)}{dx^2}\theta(t) = 0$$

Divide both sides of the equation by $u(x,t) = R(x)\,\theta(t)$

$$\Rightarrow \underbrace{\frac{\theta'(t)}{k\,\theta(t)}}_{\text{function of } t} = \underbrace{\frac{R''(x)}{R(x)}}_{\text{function of } x}$$ this must hold for $\forall\, x, t$.

$$\Rightarrow \frac{\theta'(t)}{k\,\theta(t)} = \text{constant} = -\gamma^2 = \frac{R''(x)}{R(x)}$$

The separation constant $(-\gamma^2)$ must be negative so that the $R(x)$ function is a linear combination of sines and cosines (not sinh and cosh). The former are complete and can vanish at more than one x.

Notice that 1 partial DE has become 2 ordinary DEs.

$$R''(x) + \gamma^2 R(x) = 0 \Rightarrow R(x) = A\cos(\gamma x) + B\sin(\gamma x)$$

2nd order DE will have two arbitrary constants.

$$\theta'(t) + \gamma^2 k\,\theta(t) = 0 \Rightarrow \theta(t) = C\,e^{-\gamma^2 k t}$$

1st order DE will have one arbitrary constant.

So $u(x,t) = R(x)\,\theta(t) = e^{-\gamma^2 kt}\left[\underset{\substack{\uparrow \\ \alpha \equiv AC}}{\alpha} \cos(\gamma x) + \underset{\substack{\uparrow \\ \beta \equiv BC}}{\beta} \sin(\gamma x)\right]$

The boundary condition at $x=0$ is $u(0,t)=0$

$$u(0,t) = e^{-\gamma^2 kt}\left[\alpha \overset{1}{\cancel{\cos(0)}} + \beta \overset{0}{\cancel{\sin(0)}}\right] = 0$$

$\Rightarrow \alpha = 0$ since the time exponential can't be zero.

Thus far,
$$u(x,t) = \beta e^{-\gamma^2 kt} \sin(\gamma x)$$

The boundary condition at $x=a$ is $u(a,t)=0$

$$u(a,t) = \beta e^{-\gamma^2 kt} \sin(\gamma a) = 0$$

We don't want to set $\beta = 0$ because that would give the trivial solution $u(x,t) = 0 \;\forall\, x,t$ which does satisfy the heat equation, but does not fit the initial condition at $t=0$. The time exponential is never zero, so

$$\sin(\gamma a) = 0 \Rightarrow \gamma = \frac{n\pi}{a} \quad \text{for } n = 1, 2, 3, \ldots$$
(γ is quantized)

Now
$$u(x,t) = \beta e^{-\frac{n^2\pi^2}{a^2}kt} \sin\left(\frac{n\pi x}{a}\right)$$

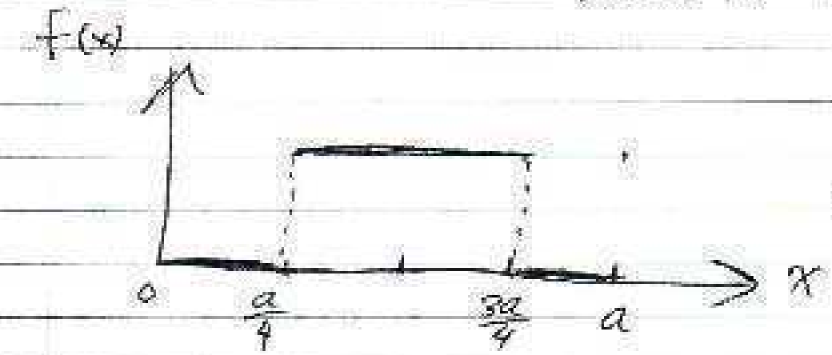
but there is a linearly independent solution for every positive integer n so the complete solution is

$$u(x,t) = \sum_{n=1}^{\infty} \beta_n \, e^{-\frac{n^2\pi^2}{a^2}kt} \sin\left(\frac{n\pi x}{a}\right)$$

the last boundary condition (initial condition) at $t=0$ can be satisfied by choosing the coefficients β_n appropriately.

$$u(x,0) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{a}\right) = f(x) = \begin{cases} 0, & 0 \le x < \frac{a}{4} \\ T_0, & \frac{a}{4} < x < \frac{3a}{4} \\ 0, & \frac{3a}{4} < x \le a \end{cases}$$

where $T_0 = 100°C$



← make this function odd and periodic

New period = $2a$.

Now use Fourier's trick to solve for β_p.

Multiply both sides of the last equation by $\sin\left(\frac{p\pi x}{a}\right)$ and integrate $\int_{x=0}^{a} \cdots dx$

and use the orthogonality of sines and cosines!

$$\frac{2}{a}\int_{x=0}^{a} \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx = \delta_{np}$$

$$f(x) = \sum_{n=1}^{\infty} \beta_n \sin\left(\frac{n\pi x}{a}\right)$$

$$\int_{x=0}^{a} \sin\left(\frac{p\pi x}{a}\right) f(x)\,dx = \sum_{n=1}^{\infty} \beta_n \underbrace{\int_{x=0}^{a} \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right) dx}_{\frac{a}{2}\delta_{np}}$$

$$= \frac{a}{2} \sum_{n=1}^{\infty} \beta_n \delta_{np} = \frac{a}{2}\beta_p$$

So now we know

$$\beta_p = \frac{2}{a} \int_{x=0}^{a} f(x) \sin\left(\frac{p\pi x}{a}\right) dx$$

$$= \frac{2}{a} \int_{x=\frac{a}{4}}^{\frac{3a}{4}} T_0 \sin\left(\frac{p\pi x}{a}\right) dx = \frac{-2T_0}{a}\left(\frac{a}{p\pi}\right) \cos\left(\frac{p\pi x}{a}\right)\Bigg|_{x=a/4}^{3a/4}$$

$$= \frac{2T_0}{p\pi}\left[\cos\left(\frac{p\pi}{4}\right) - \cos\left(\frac{3p\pi}{4}\right)\right]$$

$\beta_p = 0$ if p is even.

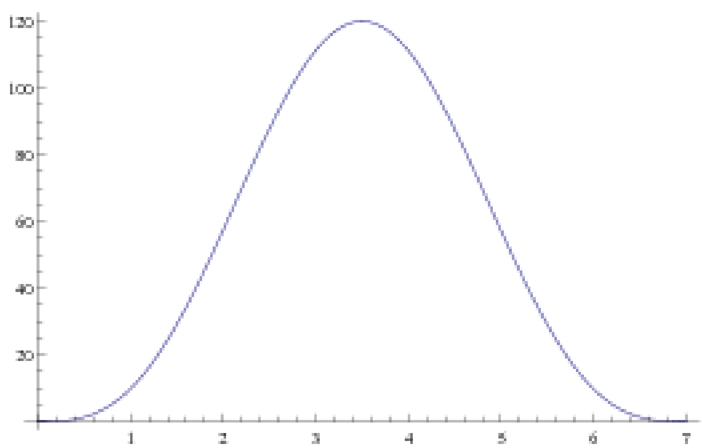
$\beta_1 = \frac{+2\sqrt{2}\,T_0}{\pi}$	$\beta_5 = \frac{-2\sqrt{2}\,T_0}{5\pi}$	$\beta_9 = \frac{+2\sqrt{2}\,T_0}{9\pi}$
$\beta_3 = \frac{-2\sqrt{2}\,T_0}{3\pi}$	$\beta_7 = \frac{+2\sqrt{2}\,T_0}{7\pi}$	$\beta_{11} = \frac{-2\sqrt{2}\,T_0}{11\pi}$

▪ PHYS 4321/7305 HW 6 - Fall 2013

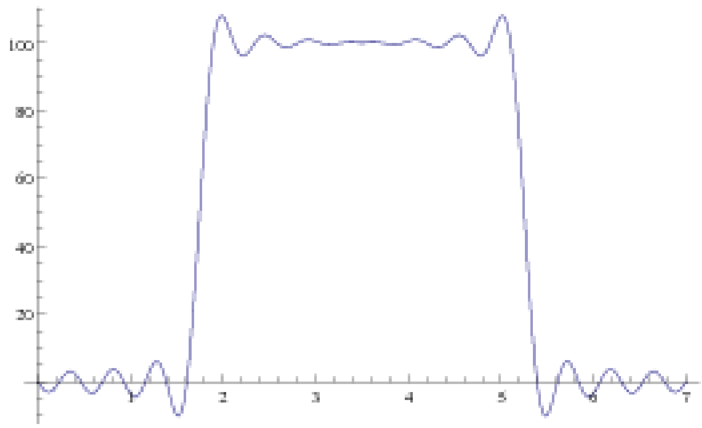
```
r = { T0 → 100,  a → 7,  k → 3};
β[p_] = 2 T0 / (p π) (Cos[p π / 4] - Cos[3 p π / 4]);
Table[{p, β[p]}, {p, 1, 12}] // TableForm
```

1	$\frac{2\sqrt{2}\,T_0}{\pi}$
2	0
3	$-\frac{2\sqrt{2}\,T_0}{3\pi}$
4	0
5	$-\frac{2\sqrt{2}\,T_0}{5\pi}$
6	0
7	$\frac{2\sqrt{2}\,T_0}{7\pi}$
8	0
9	$\frac{2\sqrt{2}\,T_0}{9\pi}$
10	0
11	$-\frac{2\sqrt{2}\,T_0}{11\pi}$
12	0

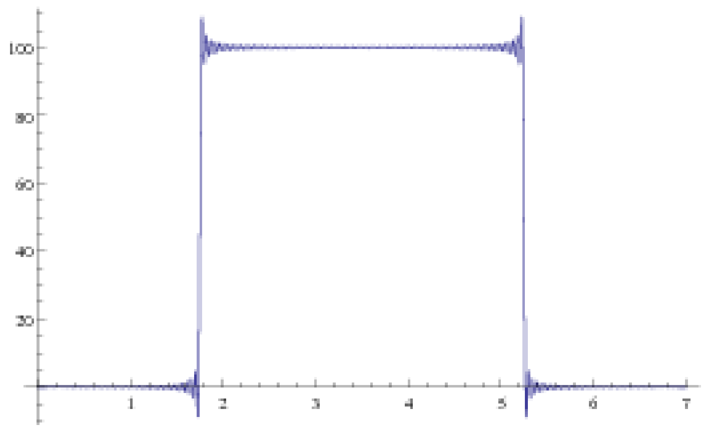
```
f[x_, n_] := Sum[β[p] Sin[p π x / a], {p, 1, n}]
Plot[f[x, 3] /. r, {x, 0, a /. r}]
```



```
Plot[f[x, 30] /. r, {x, 0, a /. r}]
```

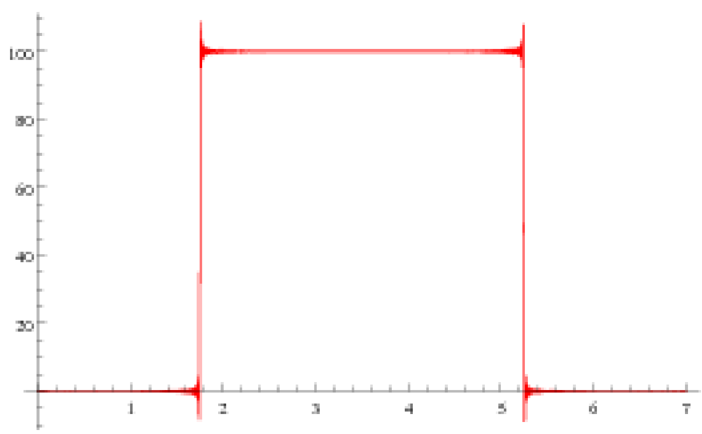


```
Plot[f[x, 300] /. r, {x, 0, a /. r}]
```

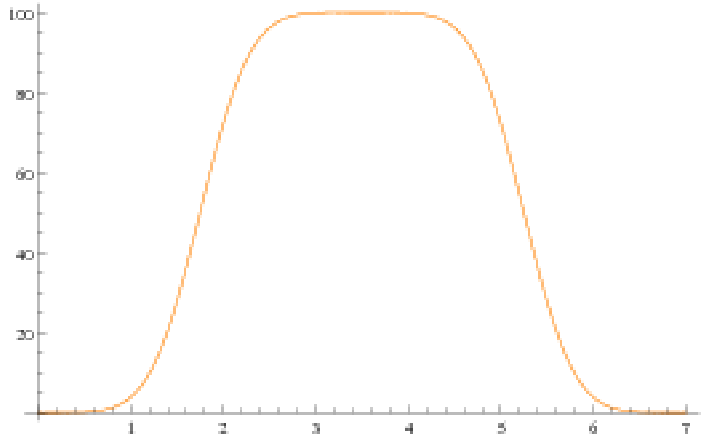


```
u[x_, t_, n_] := Sum[β[p] Exp[-p^2 π^2 k t / a^2] Sin[p π x / a], {p, 1, n}]
```

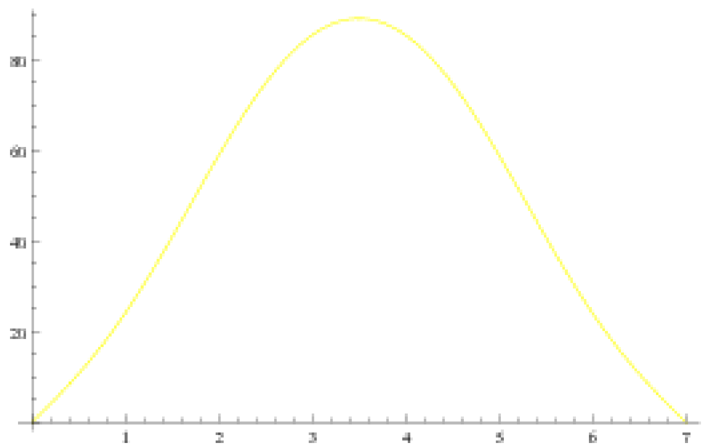
```
p1 = Plot[u[x, 0, 1000] /. r, {x, 0, a /. r}, PlotStyle → Red]
```



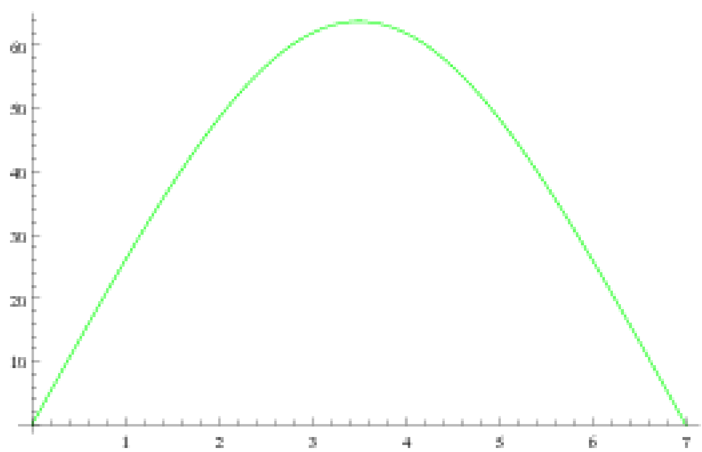
```
p2 = Plot[u[x, 0.03, 100] /. r, {x, 0, a /. r}, PlotStyle → Orange]
```



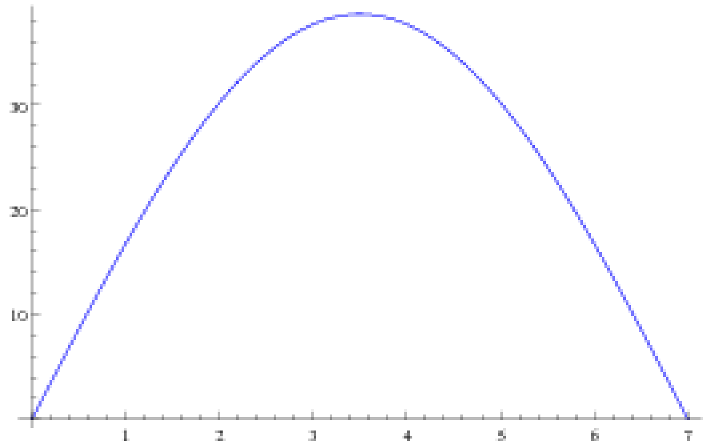

```
p3 = Plot[u[x, 0.2, 100] /. r, {x, 0, a /. r}, PlotStyle → Yellow]
```



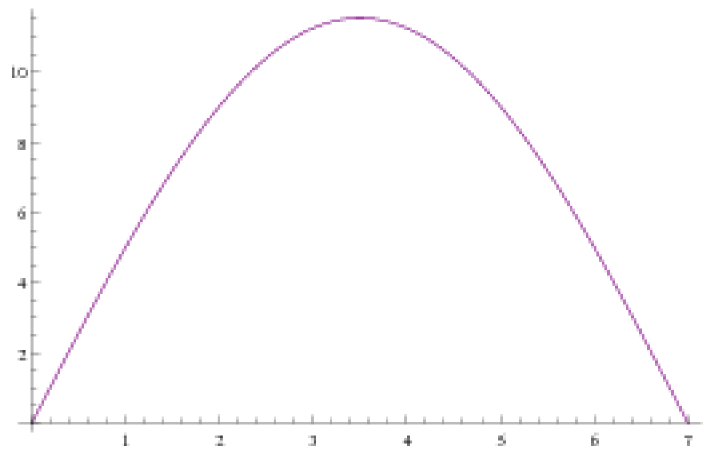

```
p4 = Plot[u[x, 0.6, 100] /. r, {x, 0, a /. r}, PlotStyle → Green]
```



```
p5 = Plot[u[x, 1.4, 100] /. r, {x, 0, a /. r}, PlotStyle → Blue]
```




```
p6 = Plot[u[x, 3.4, 100] /. r, {x, 0, a /. r}, PlotStyle → Purple]
```



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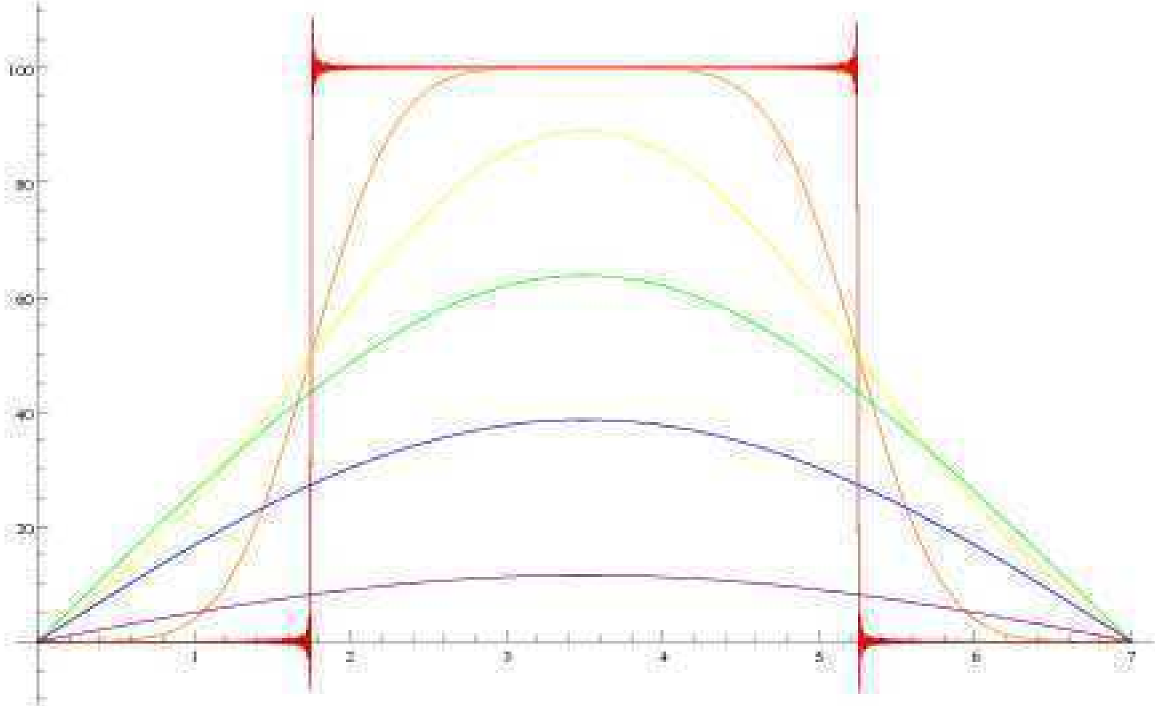
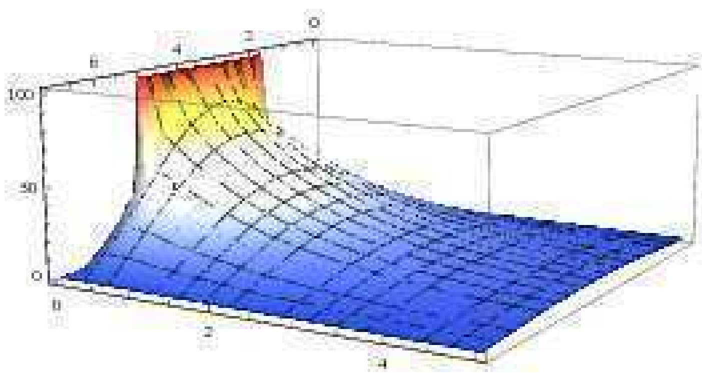
Show[p1, p2, p3, p4, p5, p6]
```




```
Plot3D[u[x, t, 300] /. r, {x, 0, a /. r}, {t, 0, 5},
 PlotRange → All, ColorFunction → (ColorData["TemperatureMap"][#3] &)]
```



Change from Cartesian to polar coordinates: $\psi(r,\varphi,t)$

but axial symmetry means $\frac{\partial\psi}{\partial\varphi} = 0$ so

$\psi(r,t) = R(r)\,T(t)$

$$\frac{\frac{\partial^2\psi}{\partial t^2} - c^2\nabla^2\psi}{\psi} = 0 \Rightarrow \underbrace{\frac{\ddot{T}(t)}{T(t)}}_{\text{function of } t} - c^2 \underbrace{\frac{\frac{1}{r}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]}{R(r)}}_{\text{function of } r} = 0$$

$$\frac{\ddot{T}(t)}{T(t)} = \text{constant} = -\omega^2 \text{ (negative for time oscillations)}$$

$$\ddot{T}(t) + \omega^2 T(t) = 0 \Rightarrow T(t) = A\cos(\omega t) + B\sin(\omega t)$$

The initial condition at $t=0$ does not mater for this problem so we can choose $A=0$. This corresponds to a flat drum head $\psi = 0$ at $t=0$.

The radial differential equation is

$$\frac{d}{dr}\left[r\,R'(r)\right] = -R(r)\frac{\omega^2}{c^2}r$$

$$\Rightarrow r\,R''(r) + R'(r) = -R(r)\frac{\omega^2}{c^2}r$$

$$\Rightarrow R''(r) + \frac{1}{r}R'(r) + \frac{\omega^2}{c^2}R(r) = 0$$

This is Bessel's differential equation for order zero.

Solutions are $R(r) = C\,J_0(\frac{\omega}{c}r) + D\,Y_0(\frac{\omega}{c}r)$

Bessel function of the first kind

Bessel function of the second kind, also Weber function.

The Weber function $Y_0(\frac{\omega}{c}r)$ blows up at $r=0$ so we must set $D=0$.

So far, our solution is

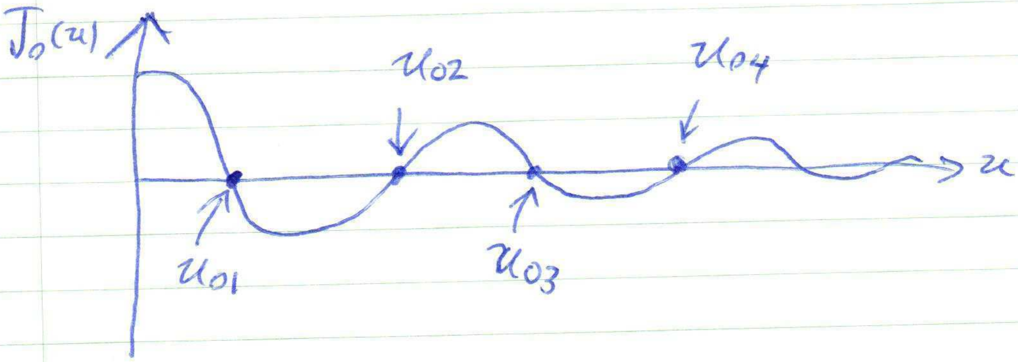
$$\psi(r,t) = R(r)\,T(t) = b\,J_0(\tfrac{\omega}{c}r)\sin(\omega t) \qquad \text{where } b \equiv BC$$

The boundary condition is that the edge of the drum head $r=a$ is fixed.

$$\psi(a,t) = 0 \quad \forall t$$

$\Rightarrow J_0(\frac{\omega}{c}a) = 0$ this is a quantization condition on ω.

$\Rightarrow \frac{\omega}{c}a =$ one of the zeros of J_0, called u_{0n} where $n = 1,2,3,\ldots$ These are tabulated.



$\omega_n \equiv \frac{c}{a} u_{0n}$ There is a linearly independent solution (normal mode of oscillation) for each n, In general

$$\Psi(r,t) = \sum_{n=1}^{\infty} b_n J_0\left(u_{0n} \frac{r}{a}\right) \sin\left(\frac{c}{a} u_{0n} t\right)$$

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(2) The lowest eigenfrequencies are

$$\omega_1 = \frac{c}{a} u_{01} = \frac{c}{a} 2.404825557695773\ldots$$

$$\omega_2 = \frac{c}{a} u_{02} = \frac{c}{a} 5.520078110286311\ldots$$

$$\omega_3 = \frac{c}{a} u_{03} = \frac{c}{a} 8.653727912911011\ldots$$