Gaussian Integrals

$$I = \int_{x=-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

not the derivative of any function that you know (probably).

plane

$$I^2 = (I)(I) = \left(\int_{x=-\infty}^{+\infty} e^{-x^2} dx\right)\left(\int_{y=-\infty}^{+\infty} e^{-y^2} dy\right) = \iint_{x\,y=-\infty}^{+\infty} e^{-(x^2+y^2)} \underbrace{dx\,dy}_{\text{area}}$$

Cartesian → Polar

$$I^2 = \int_{r=0}^{\infty}\int_{\varphi=0}^{2\pi} e^{-r^2} r\,dr\,d\varphi = \underbrace{\left(\int_{\varphi=0}^{2\pi} d\varphi\right)}_{2\pi}\left(\int_{r=0}^{\infty} e^{-r^2} r\,dr\right)$$

plane

$$I^2 = (2\pi)\left(-\frac{1}{2}e^{-r^2}\right)\Bigg|_{r=0}^{\infty} = 2\pi\underbrace{\left[-\frac{1}{2}e^{-\infty} - \left(-\frac{1}{2}\right)e^{0}\right]}_{\frac{1}{2}} = \pi$$

$$I^2 = \pi \quad \Longrightarrow \quad I = \sqrt{\pi}$$

$$\int_{y=-\infty}^{+\infty} e^{-y^2}\,dy = \sqrt{\pi}$$

change variables

$$y^2 = a x^2$$

$$y = \sqrt{a}\,x$$

$$dy = \sqrt{a}\,dx$$

$y \to \infty$

$x \to \infty$

$y \to -\infty$

$x \to -\infty$

$$\int_{x=-\infty}^{+\infty} e^{-ax^2} \sqrt{a}\,dx = \sqrt{\pi}$$

$$\int_{x=-\infty}^{+\infty} e^{-ax^2}\,dx = \sqrt{\frac{\pi}{a}}$$

$\frac{d}{da}$ both sides

Differentiate both sides w.r.t. a

$$-\int_{x=-\infty}^{+\infty} x^2 e^{-ax^2} = \sqrt{\pi}\,\frac{d}{da} a^{-1/2} = \sqrt{\pi}\left(-\frac{1}{2}\right) a^{-3/2}$$

$$\int_{x=-\infty}^{+\infty} e^{-ax^2} x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$$

what about odd?

$$\int_{-\infty}^{+\infty} e^{-ax^2} x^{2n}\,dx = ?$$

even

$$\int_{-\infty}^{+\infty} e^{-ax^2} x^{(2n+1)}\,dx = 0$$

Fourier Transform

$$\tilde{f}(k) = c(k) = \frac{1}{\sqrt{2\pi}} \int_{x=-\infty}^{+\infty} A e^{-\frac{x^2}{8}} e^{-ikx} dx$$

$$= \frac{A}{\sqrt{2\pi}} \int_{x=-\infty}^{+\infty} e^{-\left(\frac{x^2}{8} + ikx\right)} dx$$

Complete the square:

$$(ax+b)^2 = a^2x^2 + 2abx + b^2$$

$$\frac{x^2}{8} \quad + ikx$$

$$a^2 = \frac{1}{8} \qquad 2ab = ik \Rightarrow b = \frac{ik}{2a}$$

$$b^2 = \frac{i^2k^2}{4a^2} = \underline{-2k^2}$$

$$c(k) = \frac{A}{\sqrt{2\pi}} \int_{x=-\infty}^{+\infty} e^{-\left(\frac{x^2}{8} + ikx - 2k^2\right)} dx \cdot e^{-2k^2}$$

$$= \frac{A}{\sqrt{2\pi}} \underbrace{\int_{x=-\infty}^{+\infty} e^{-(ax+b)^2} dx}_{\text{Gaussian integrals}} \cdot e^{-2k^2}$$

change variables $y = ax+b$

$dy = a\,dx$

$$c(k) = B e^{-2k^2}$$