Nonlinear Simple Pendulum

$$\ddot{\theta}(t) = -\frac{g}{\ell}\sin[\theta(t)]$$

If $[\dot{\theta}(t)]^2 = \frac{2g}{\ell}\left\{\cos[\theta(t)] - \cos[\underbrace{\theta_0}_{\text{constant}}]\right\}$ then

$$\frac{d}{dt}[\dot{\theta}(t)]^2 = 2\dot{\theta}(t)\ddot{\theta}(t) = -\frac{2g}{\ell}\sin[\theta(t)]\,\dot{\theta}(t)$$

$$\Rightarrow \ddot{\theta}(t) = -\frac{g}{\ell}\sin[\theta(t)] \quad \text{when } \dot{\theta}(t) \neq 0 \;\forall t$$

$$\dot{\theta}(t) = \sqrt{\frac{2g}{\ell}[\cos\theta - \cos\theta_0]} = \frac{d\theta}{dt}$$

$$T = 4\int_{t=0}^{t(\theta_0)} dt = 4\int_{\theta=0}^{\theta_0} \frac{d\theta}{\sqrt{\frac{2g}{\ell}[\cos\theta - \cos\theta_0]}}$$

substitute $\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$

$$T = 4\sqrt{\frac{\ell}{2g}}\int_{\theta=0}^{\theta_0} \frac{d\theta}{\sqrt{2\sin^2\left(\frac{\theta_0}{2}\right) - 2\sin^2\left(\frac{\theta}{2}\right)}} = 2\sqrt{\frac{\ell}{g}}\int_{\theta=0}^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}}$$

Change of variable: $\sin\left(\frac{\theta}{2}\right) \equiv \sin\left(\frac{\theta_0}{2}\right) \sin\varphi$

$$d\left[\sin\left(\frac{\theta}{2}\right)\right] = \frac{1}{2}\cos\left(\frac{\theta}{2}\right) d\theta = \sin\left(\frac{\theta_0}{2}\right)\cos\varphi \, d\varphi$$

$$T = 2\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\pi/2} \frac{d\varphi \; 2\cos\varphi \, \sin\left(\frac{\theta_0}{2}\right)}{\cos\left(\frac{\theta}{2}\right)\sqrt{\sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta_0}{2}\right)\sin^2\varphi}}$$

$$T = 2\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\pi/2} \frac{d\varphi \; 2\cos\varphi}{\cos\left(\frac{\theta}{2}\right)\sqrt{1-\sin^2\varphi}} = 4\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\pi/2} \frac{d\varphi}{\cos\left(\frac{\theta}{2}\right)}$$

$$T = 4\sqrt{\frac{l}{g}} \int_{\varphi=0}^{\pi/2} \frac{d\varphi}{\sqrt{1 - \sin^2\left(\frac{\theta_0}{2}\right)\sin^2\varphi}}$$

Binomial Expansion (convergent for $x < 1$)

$$\frac{1}{\sqrt{1-x}} = 1 + \frac{1}{2}x + \frac{1}{2}\cdot\frac{3}{4}x^2 + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}x^3 + \dots$$

$$x = \sin^2\left(\frac{\theta_0}{2}\right)\sin^2\varphi < 1$$

$$T = 4\sqrt{\frac{\ell}{g}} \int_{\varphi=0}^{\pi/2} d\varphi \left[1 + \frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\sin^2\varphi + \frac{1}{2}\cdot\frac{3}{4}\sin^4\left(\frac{\theta_0}{2}\right)\sin^4\varphi + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\sin^6\left(\frac{\theta_0}{2}\right)\sin^6\varphi + \cdots \right]$$

Now $\int_{\varphi=0}^{\pi/2} \sin^2\varphi = \frac{\pi}{4}$ and $\int_{\varphi=0}^{\pi/2} \sin^4\varphi = \frac{3\pi}{16}$

$$T = 4\sqrt{\frac{\ell}{g}} \left[\frac{\pi}{2} + \frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right)\frac{\pi}{4} + \frac{1}{2}\cdot\frac{3}{4}\sin^4\left(\frac{\theta_0}{2}\right)\frac{3\pi}{16} + \cdots \right]$$

$$T = \underset{\substack{\uparrow \\ \text{linear} \\ \text{result}}}{2\pi\sqrt{\frac{\ell}{g}}} \Bigg[1 + \underbrace{\left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\theta_0}{2}\right) + \left(\frac{1}{2}\frac{3}{4}\right)^2 \sin^4\left(\frac{\theta_0}{2}\right) + \cdots}_{\text{non-linear corrections}} \Bigg]$$

This period is written as an expansion in $\sin^2\left(\frac{\theta_0}{2}\right)$.

To compare with Marion, where the expansion is in $\frac{\theta_0}{2}$,

use $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - + \cdots$