Final

emphasize 2nd half of

3 main problems w/ lots of parts for more credit

Boundary Conditions depend on all 3 coordinates (ρ, φ, z)

Laplace: $\nabla^2 \Phi(\rho, \varphi, z) = 0$

Assume: $\Phi(\rho, \varphi, z) = R(\rho)\, F(\varphi)\, Z(z)$

$$\frac{\nabla^2 \Phi}{\Phi} = 0 = \underbrace{\frac{(\rho R'(\rho))'}{\rho \boxed{R(\rho)}} + \frac{1}{\rho^2}\frac{F''(\varphi)}{F(\varphi)}}_{f(\rho, \varphi)} + \underbrace{\frac{Z''(z)}{Z(z)}}_{G(z)}$$

b/c you're dividing by Φ

↑ we didn't successfully separate variables

1st separation constant is C

equal & opposite

$$\frac{(\rho R')'}{\rho R} + \frac{1}{\rho^2}\frac{F''(\varphi)}{F(\varphi)} = \boxed{C} \quad \Bigg\| \quad \frac{Z''(z)}{Z(z)} = \boxed{-C}$$

depending on sign of C, you'll get exp's or cos/sines

now separate this:

don't forget this term

$$\underbrace{\frac{\rho(\rho R')'}{R} - \boxed{\rho^2 C}}_{h(\rho)} + \underbrace{\frac{F''(\varphi)}{F(\varphi)}}_{j(\varphi)} = 0$$

2nd separation constant is K

$$\frac{\rho(\rho R')'}{R} + \rho^2 C = K \quad \Bigg\| \quad \frac{F''(\varphi)}{F(\varphi)} = -K$$

depending on sign of K, you get exps or cos/sines

<u>Now you have these 3 Equations</u>

$Z''(z) + C Z(z) = 0$ — Completely general cylindrical

$F'(\varphi) + KF(\varphi) = 0$

$$\frac{\rho(\rho R')'}{R} - \rho^2 C = K$$

Now you have to consider initial & boundary conditions:

Special case: full cylinder → $0 \le \varphi < 2\pi$

∴ the first 2 equations must be periodic in 2π

∴ K must be some positive integer squared

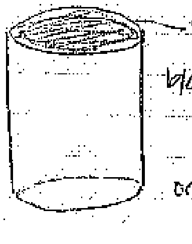
$K = n^2$, $n = 0, 1, 2, 3, \ldots$

∴ $F''(\varphi) + n^2 F(\varphi) = 0$

$F(\varphi) = A\cos(n\varphi) + B\sin(n\varphi)$ ← completeness b/c it's a full cylinder

Which implies:

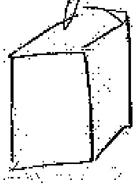
$$\frac{\rho(\rho R')'}{R} - \rho^2 C = n^2$$



b/c z is fixed ∴ ρ & φ require oscillatory functions & growth/decay in z

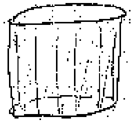
b/c The constants are tied by a negative sign ~ one must oscilate & one must grow/decay

Recall from Cartesian System:



$f''(x) + \alpha^2 f(x) = 0$
$g''(y) + \beta^2 g(y) = 0$ } oscillatory complete
$h''(z) - (\alpha^2+\beta^2) h(z) = 0$ } growth decay

& Back to Cylindrical Example:



$W(z, \varphi)$
oscillatory in z, φ $c<0$
and growth/decay in ρ $c>0$

Case #1 : $C=0$

$Z''(z) = 0 \Longrightarrow Z(z) = a_1 z + a_2$ where a_1 & a_2 are constants

$F(\varphi)$ is same as before : $F(\varphi) = A\cos(n\varphi) + B\sin(n\varphi)$

$\frac{\rho(\rho R')'}{R} + 0 = n^2$ $\underbrace{R(\rho) = a_3 \rho^n + a_4 \rho^{-n}}_{\text{remember from Monday before break, we solved a problem like this}}$

Case #2 $C = -\alpha^2$

$Z''(z) - \alpha^2 Z(z) = 0$
$Z(z) = a_5 e^{\alpha z} + a_6 e^{-\alpha z}$
or also $Z(z) = a_7 \sinh(\alpha z) + a_8 \cosh(\alpha z)$

$F(\varphi)$ is the same : $A\cos(n\varphi) + B\sin(n\varphi)$

$\frac{1}{\rho}(\rho R')' + \left(\alpha^2 - \frac{n^2}{\rho^2}\right)R = 0$ same ρ equation simply rewritten

$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho \frac{dR(\rho)}{d\rho}\right] + \left(\alpha^2 - \frac{n^2}{\rho^2}\right)R(\rho) = 0$ Bessel's Equation

Solutions to Bessel's Equation:

$R(\rho) = a_9 J_n(\alpha\rho) + a_{10} N_n(\alpha\rho)$

2 separation constants (α^2 & n^2)

order

always take $\alpha\rho$, b/c ρ has unit length so you always need the α ($\frac{1}{L}$) so that the units cancel nicely

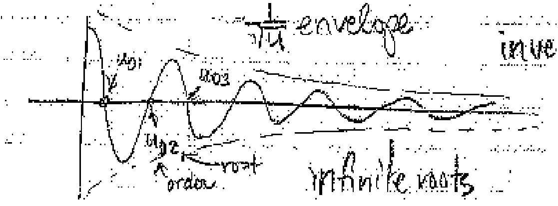
J_n Bessel function of the first type.

$N_n = Y_n$, called either Bessel function of 2nd type or Weber function

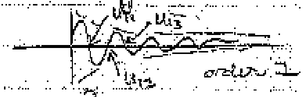
J_n & N_n oscillate just like sines & cosines

Mathematica plotting of Bessel Functions:

Plot [BesselJ [order, u], {u, range, range}]

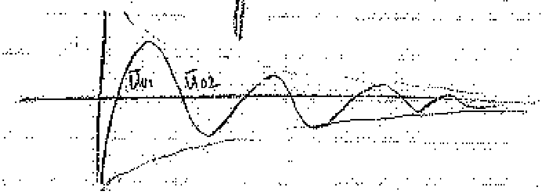


inverse quadratic decay ~not quite exponential



only J_0 starts at a nonzero value @ zero (1)

* All Weber functions @ 0 are negative infinity



∴ if $\rho = 0$ is part of the problem, you must exclude Weber functions (N_n & Y_n)

"Anything that wiggles is complete"

$$V(\rho) = \sum_{n=0}^{\infty}\left[A_n J_n(\alpha\rho) + B_n N_n(\alpha\rho)\right]$$

another pair of basis functions, like sines/cosines

What if you flip sign of C? $C = +\alpha^2$

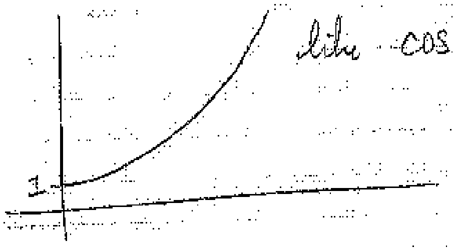
$Z(z) = a_{11}\cos(\alpha z) + a_{12}\sin(\alpha z)$

$F(\varphi)$ stays the same! $F(\varphi) = A\cos(n\varphi) + B\sin(n\varphi)$

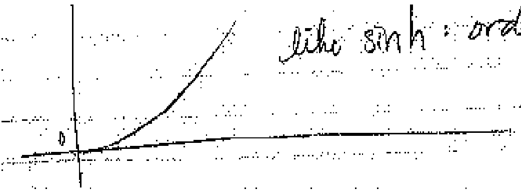
$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dR}{d\rho}\right] \underset{\uparrow}{-} \left(\alpha^2 + \frac{n^2}{\rho^2}\right)R = 0$$

factored out the negative sign

$R(\rho) = a_{13} I_n(\alpha\rho) + a_{14} K_n(\alpha\rho)$

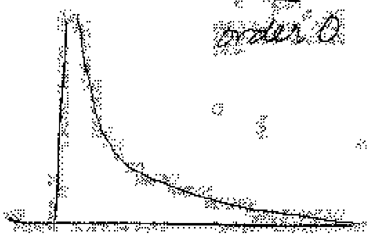


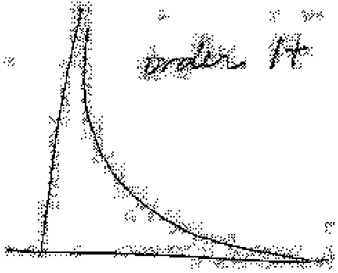
like cosh : order 0



like sinh : order 1+

K_n Bessel functions:



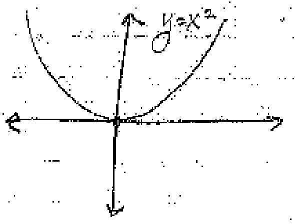


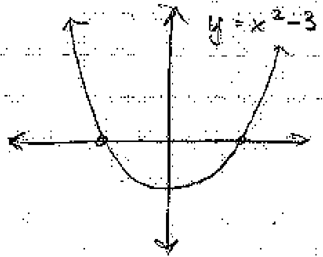
If you don't have ∞ & you don't have zero, you can have I_n & K_n together

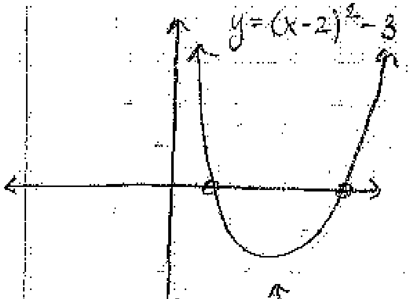
~Introduction to Complex Numbers~

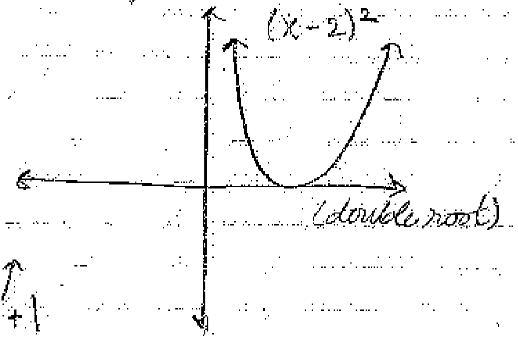
Simplest equation you can't solve w/o i:

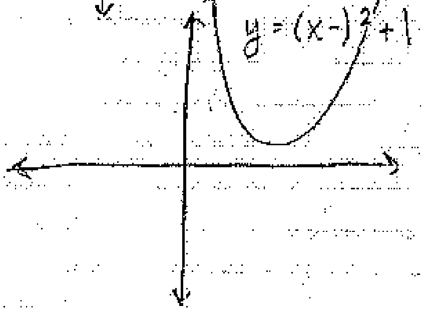
$x^2+1=0$











roots are complex conjugates (2)

Givolamo Cardano

$ax^2+bx+c=0$ $\quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

"General Quadratic Solution"

~next page

Cardano discovered: (1501 → 1576)

$x^3-px-q=0$

$$x=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}-\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}-\frac{p^3}{27}}}$$

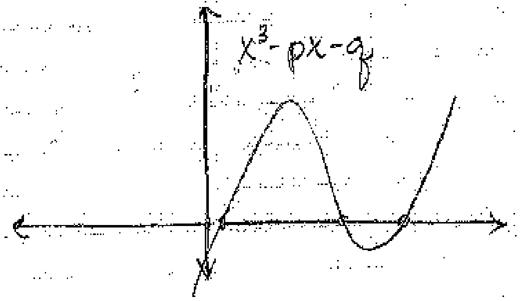
*All good, except that if you got a negative under the radical, there were problems w/o i

Ex. $p=15$ $q=4$

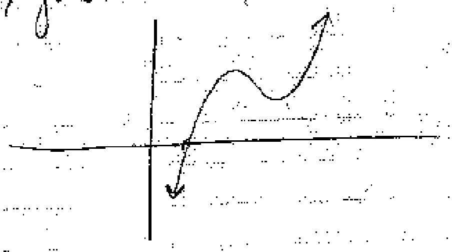
$$x=\sqrt[3]{2-\sqrt{-121}}-\sqrt[3]{-2-\sqrt{121}}$$

one of the 3 answers is actually real b/c the i's cancel out, but you still need i to get to the answers.

(Actually, all 3 of the solutions for x are real: $x=4$, $-2+\sqrt{3}$, $-2-\sqrt{3}$



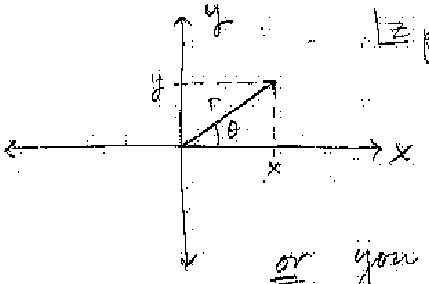
But, you could also have:



and thus only one real root & the others are imaginary

Complex Numbers

$\sqrt{-1} = \pm i$



$\boxed{z}$ plane

$x = Re(z)$
$y = Im(z)$
$z = x + iy$
Cartesian

or you could write it in Polar
$z = re^{i\theta}$

Taylor Expansion about zero

$$e^x = e^0 + x \left.\frac{de^x}{dx}\right|_{x=0} + \frac{x^2}{2!} \left.\frac{d^2e^x}{dx^2}\right|_{x=0} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

defined by Taylor Expansion

$$e^z \equiv 1 + z + \frac{z^2}{2!} + \ldots \rightsquigarrow x^2 - y^2 + 2ixy \ldots$$
$$= 1 + (x+iy) + \frac{(x+iy)^2}{2!} + \ldots$$

$$\boxed{\begin{array}{l} i^1 = i \\ i^2 = -1 \\ i^3 = -i \\ i^4 = +1 \end{array}}$$ & then it repeats...

$$e^{x+iy} = e^x[\underbrace{\cos y}_{\text{all even powers}} + i\underbrace{\sin y}_{\text{all odd powers}}]$$ & a derivation of Euler's Formula

$z = re^{i\theta}$

↳ magnitude of z $|z| = \sqrt{x^2+y^2}$

$z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta}$

$$\boxed{z^n = r^n e^{in\theta}}$$

represent in complex form

$\sqrt[3]{1} = \sqrt[3]{1e^{\{0, 2\pi, 4\pi, 6\pi \ldots\}i}}$

↳ any of these would work

$r^3 = 1 \Rightarrow r = \sqrt[3]{1} = 1$

$e^{0} = e^{3i\theta} \Rightarrow \theta = 0$

$e^{i2\pi} = e^{3i\theta} \Rightarrow \theta = \frac{2\pi}{3}$

$e^{i4\pi} = e^{3i\theta} \Rightarrow \theta = \frac{4\pi}{3}$

$e^{i6\pi} = e^{3i\theta} \Rightarrow \theta = 2\pi = 0$ & it starts repeating

$z^3 = 1 \Rightarrow z = re^{i\theta} \quad 1e^{i0} = \underline{\underline{1}}$

$z = re^{i\theta} = 1e^{i\frac{2\pi}{3}} = 1(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$

using Euler's formula

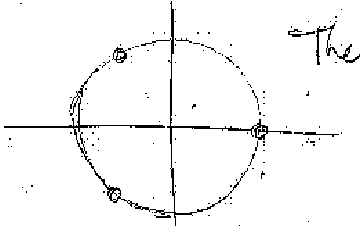
$z = \cos(120°) + i\sin(120°)$

$= \boxed{\frac{1}{2} + i\sqrt{3}/2}$

$z = re^{i\theta} = 1e^{i4\pi/3} = 1(\cos(4\pi/3) + i\sin(4\pi/2))$

$= \cos(240°) + i\sin(240°) = \boxed{\frac{1}{2} - \frac{i\sqrt{3}}{2}}$

Then plot them on the unit circle:



The center of mass of these pts is zero.

If there were 4 roots, they would be:
1, i, -1 & $-i$

$$i^i = \left(1e^{i\pi/2}\right)^i = 1e^{i^2 \overset{\bigcirc}{\pi/2}} = 1e^{-\pi/2} = \frac{1}{e^{\pi/2}} = \frac{1}{\sqrt{e^{\pi}}}$$

$= .2078\ldots$

but, you have an infinite # of values to plug in here that would work. $5\pi/2$

$$i^i = e^{i(2\pi n + \frac{\pi}{2})}$$

Complex Conjugate:

$i \to -i$

$z^* = x - iy$

$z^* = re^{-i\theta}$

$W = z^z$

Branch cut