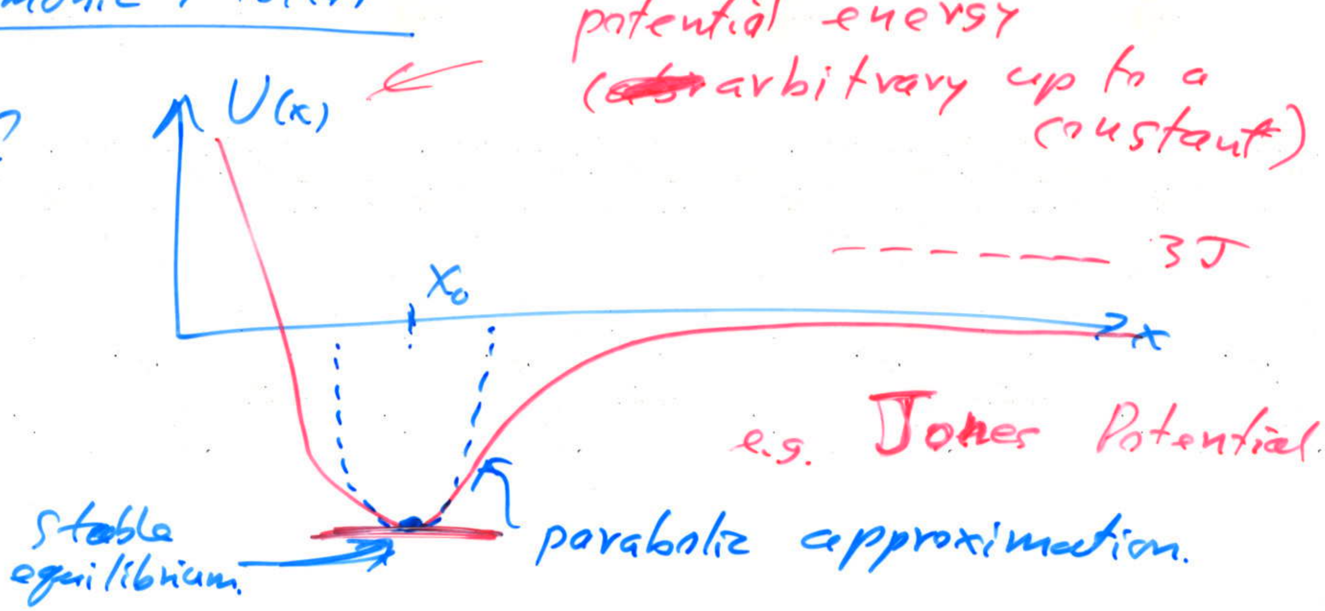
Harmonic Motion

Why?



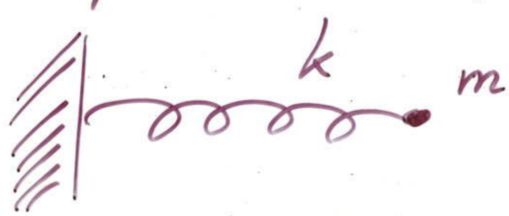
Taylor Expand $U(x)$ around the minimum

$$U(x) = \underbrace{U(x_0)}_{\text{constant}} + \underbrace{\cancel{(x-x_0)\left.\frac{dU}{dx}\right|_{x=x_0}}}_{\text{minimum}} \rightarrow 0 + \frac{1}{2!}(x-x_0)^2 \underbrace{\left.\frac{d^2U}{dx^2}\right|_{x=x_0}}_{\text{if positive concavity}} + \dots$$

$$U(x) \approx \frac{1}{2}k(x-x_0)^2 \qquad k>0$$

same k in Hooke's law $F=-kx$

Simple Harmonic Motion



Newton's 2nd Law

$$\vec{F} = -k\vec{x}$$

one dimension: $F=-kx$ restoring force

$$F = -\frac{dU}{dx}$$

$$ma = -kx$$

$$m\frac{d^2x(t)}{dt^2} = -k\,x(t)$$

$$\ddot{x}(t) + \underbrace{\frac{k}{m}}_{\omega_0^2} x(t) = 0$$

2^{nd} order, linear in x homogeneous ordinary difl. eq.

$$\ddot{x}(t) + \omega_0^2 x(t) = 0$$

solutions (2 arbitrary constants)

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$
$$= \boxed{C e^{i\omega_0 t} + D e^{-i\omega_0 t}}$$
$$= E\sin(\omega_0 t + F)$$
$$= G\cos(\omega_0 t + H)$$

Guess $x(t) = Ae^{rt}$

$$\dot{x}(t) = rAe^{rt}$$

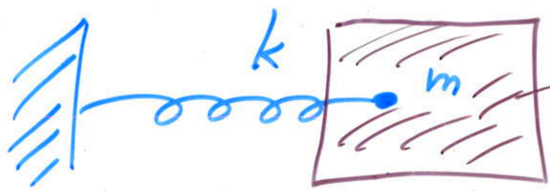
$$\ddot{x}(t) = r^2 Ae^{rt}$$

Substitute: $r^2 A e^{rt} + \omega_0^2 A e^{rt} = 0$

$$A \cdot e^{rt} \cdot (r^2 + \omega_0^2) = 0 \Longrightarrow r^2 = -\omega_0^2$$

$$r_{\pm} = \pm i\omega_0$$

Damped Simple Harmonic Motion (SHM)



fluid with viscous drag coefficient b

Aside: Surface-Surface friction $\propto -\hat{v}$

Fluid-viscous drag $\vec{F} = -b\vec{v}$

Air drag $F = -cv^2\hat{v}$

Newton's 2^{nd} Law

$$\vec{F} = -k\vec{x} - b\vec{v} \rightarrow \text{one dimension}$$

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = 0$$

$$\ddot{x}(t) + \underbrace{\frac{b}{m}}_{2\beta}\dot{x}(t) + \underbrace{\frac{k}{m}}_{\omega_0^2}x(t) = 0$$

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$$

2nd order, linear homogeneous

Guess: $x(t) = Ae^{rt}$ / $\dot{x}(t) = rAe^{rt}$ / $\ddot{x}(t) = r^2Ae^{rt}$

Substitute:

$$\underbrace{(r^2 + 2\beta r + \omega_0^2)}_{\text{quadratic equation}}Ae^{rt} = 0$$

quadratic equation
- solve for r

$$r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Complementary solution:

$$x_c(t) = A_{(+)}e^{r_{(+)}t} + A_{(-)}e^{r_{(-)}t}$$

$$= \left(A_{(+)}e^{+\sqrt{\beta^2-\omega_0^2}\,t} + A_{(-)}e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)\underbrace{e^{-\beta t}}_{\text{damping envelope}}$$

damping envelope

Three sub cases:

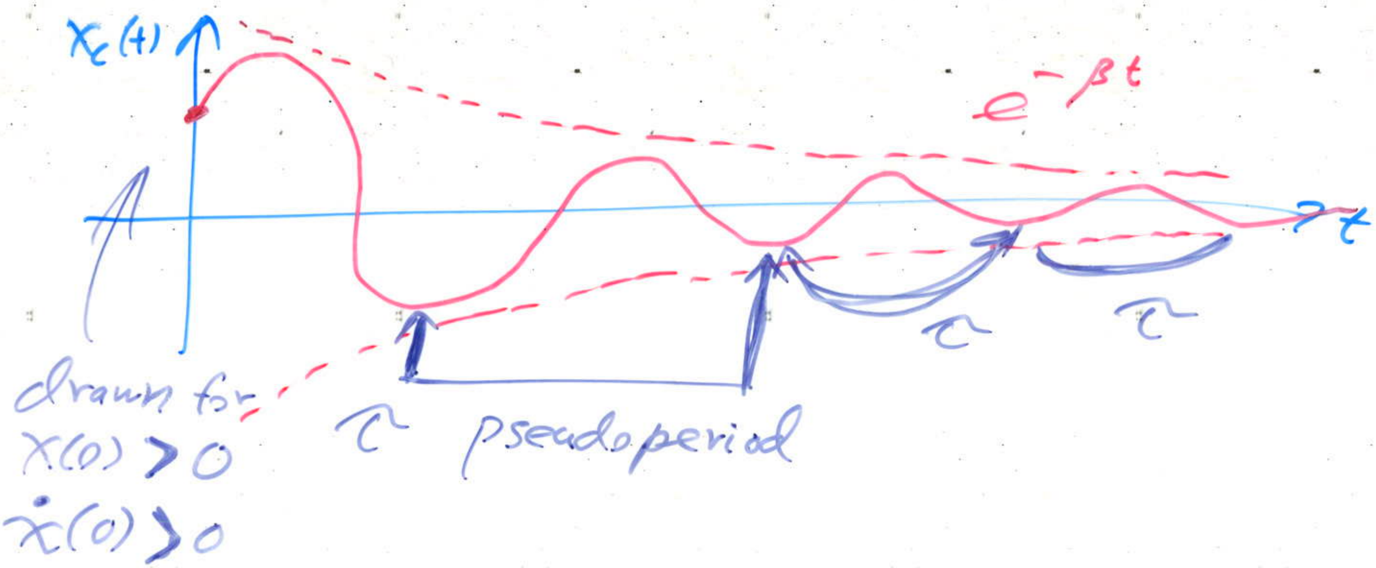
① under damped motion: $\beta < \omega_0$

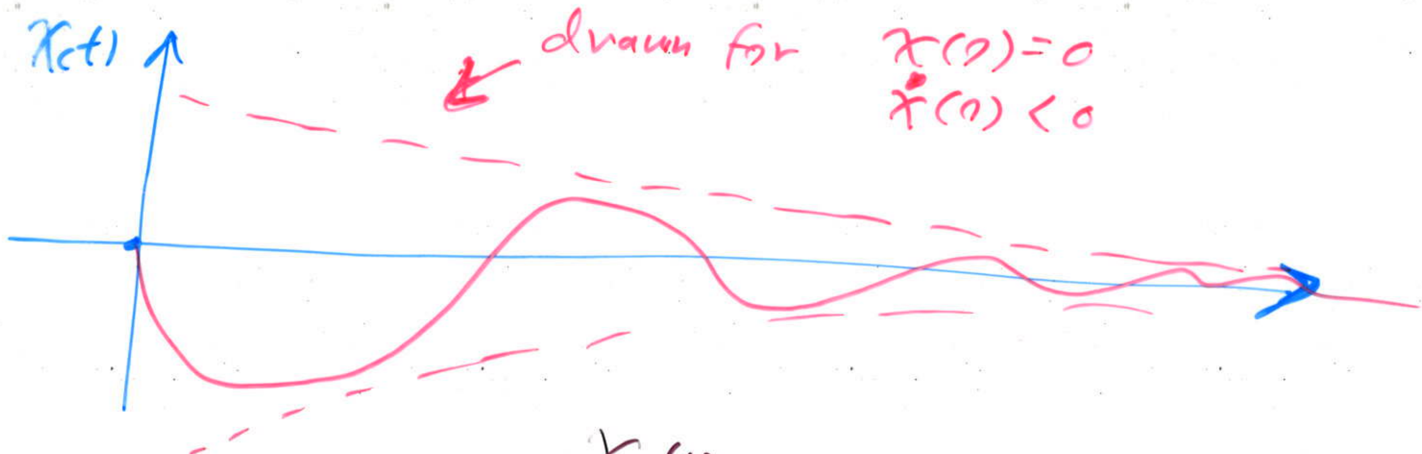
define $\omega_1^2 = -\beta^2 + \omega_0^2 > 0$ $\quad \omega_1 \in \mathbb{R}$eal

$$\pm\sqrt{\beta^2 - \omega_0^2} = \pm i\omega_1 t$$

$$x_c(t) = e^{-\beta t}\left[A_{(+)}e^{+i\omega_1 t} + A_{(-)}e^{-i\omega_1 t}\right]$$

$$= e^{-\beta t}\left[B_1 \cos(\omega_1 t) + B_2 \sin(\omega_1 t)\right]$$





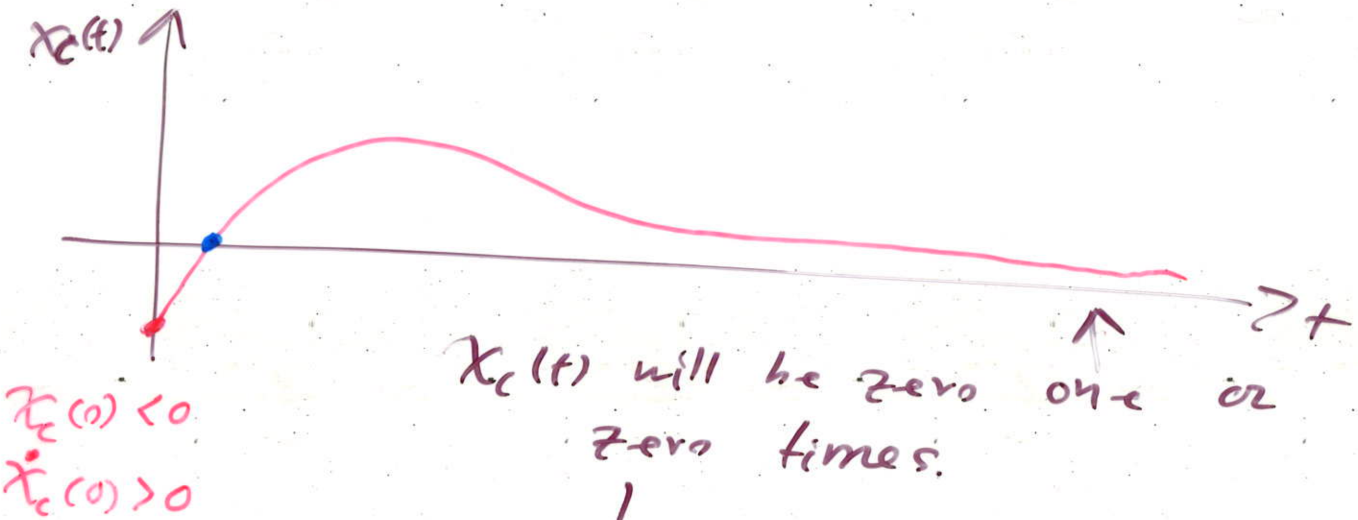
$x_c(t) = 0$ an infinite number of times.

e.g. <u>music</u>, bungee jumping.

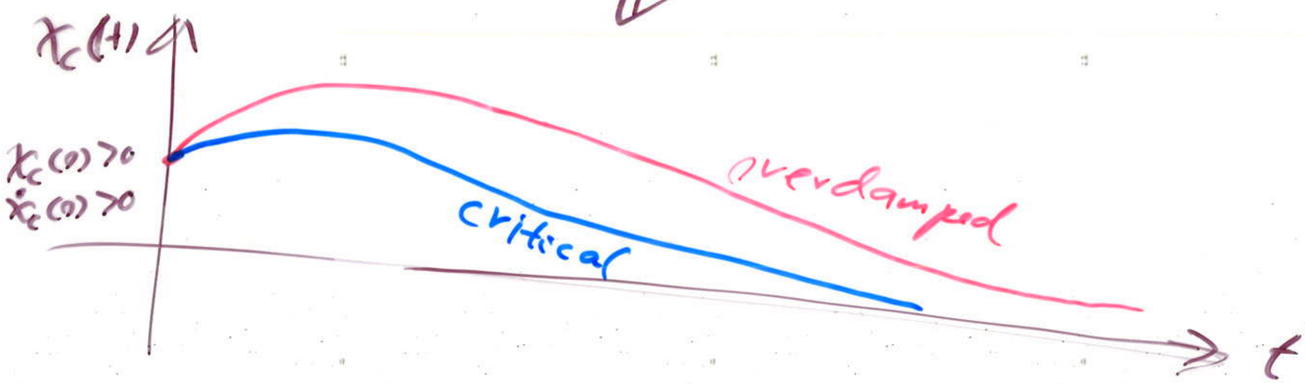
② Overdamped Motion $\beta > \omega_0$

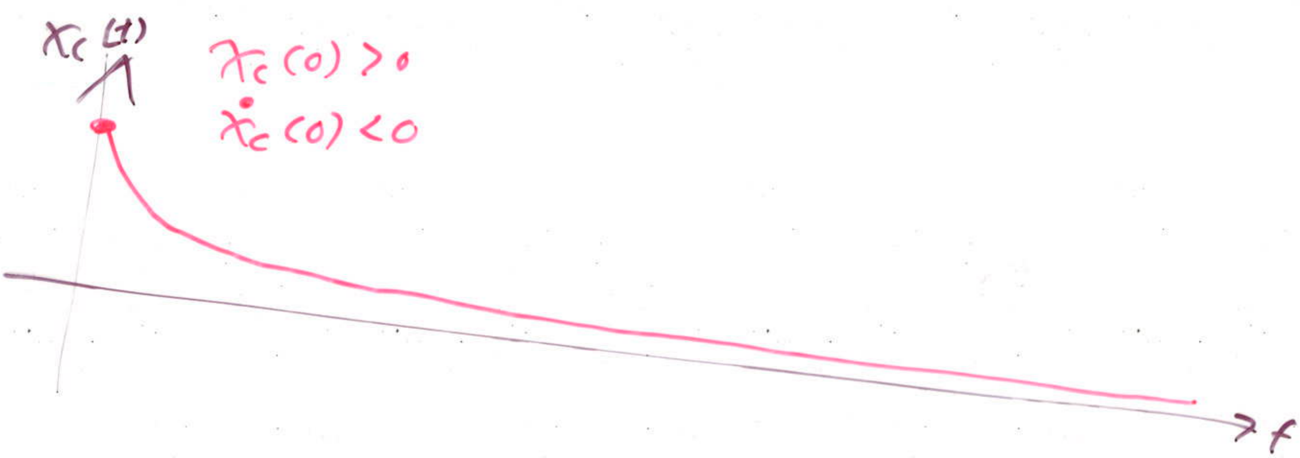
$$\sqrt{\beta^2 - \omega_0^2} \equiv \omega_2 > 0$$

$$x_c(t) = e^{-\beta t}\left[A_{(+)}\,e^{+\omega_2 t} + A_{(-)}\,e^{-\omega_2 t}\right]$$



$x_c(t)$ will be zero one or zero times.





③ Critical damping $\beta = \omega_0$ / $\frac{\gamma}{2} = \beta$

e.g. Shock absorbers.

$$x_c(t) \overset{?}{=} e^{-\beta t}\left[A_{(+)} e^{0} + A_{(-)} e^{0}\right]$$

not linearly independent.

need a second linearly independent solution.

$\Longrightarrow$ multiply 1^{st} solution by powers of t.

$$x_c(t) = A_1 e^{-\beta t} + A_2\, t\, e^{-\beta t}$$ check A_2 term

$$\dot{x}_c(t) = A_2\left[e^{-\beta t} + t(-\beta)e^{-\beta t}\right] = A_2 e^{-\beta t}\left[1-\beta t\right]$$

$$\ddot{x}_c(t) = A_2 e^{-\beta t}\left[-2\beta + \beta^2 t\right]$$

substitute

$$\ddot{x}_c(t) + 2\beta\dot{x}_c(t) + \omega_0^2 x_c(t) = 0$$

$$A_2 e^{-\beta t}\left[(-2\beta + \beta^2 t) + 2\beta(1-\beta t) + \underset{=\beta^2}{\omega_0^2}\, t\right] = 0 \checkmark$$

Why does multiplying one solution by t work?

start with $x_c(t) = e^{-\beta t}\left[A_{(+)} e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_{(-)} e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$

take limit $\beta \to \omega_0 \Longrightarrow e^{-xt}$ for small x

Taylor expansion: $1 - xt + \ldots$

$$x_c(t) \xrightarrow[\beta \to \omega_0]{\lim} e^{-\beta t}\left[A_{(1)} 1 - A_{(2)} t\right]$$

$$= A_{(1)} e^{-\beta t} + A_{(2)} t e^{-\beta t}$$

Use Wronskian to check linear independence.

Driven Damped Harmonic Motion.