Group Theory – study of symmetries

symmetry – an operation that leaves the system unchanged.

A group is a set with a binary operation.

That satisfies 3 axioms

A binary operation on a set is a rule ($\cdot$) which assigns to each ordered pair of elements of the set some element of the set.

e.g. the set $G = \{a, b, c\}$

$a \cdot b = c$, $b \cdot a = a$, $c \cdot c = b$

1) exactly one element is assigned to each ordered pair

2) for each ordered pair, the element assigned to it is in the Group (closure)

Examples of sets:

$\mathbb{Z}$ is the set of all integers ... $-2, -1, 0, +1, +2$ ...

$\mathbb{Z}^+$ " " positive integers $1, 2, 3$ ...

$\mathbb{Q}$ " " rational numbers $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}$ ...

$\mathbb{Q}^+$ positive " "

$\mathbb{R}$ " " real

$\mathbb{R}^+$ positive real

$\mathbb{C}$ complex numbers

Standard Model Particles $SU(3)_c \times SU(2)_L \times U(1)$

$\mathbb{Z}^+$: $a \cdot b = \min(a,b)$ e.g. $2 \cdot 5 = 2$, $3 \cdot 3 = 3$ ✓

valid binary operation

$\mathbb{Z}^+$: $a \cdot b = a$ e.g. $2 \cdot 5 = 2$, $5 \cdot 2 = 5$, $3 \cdot 3 = 3$ ✓

$\mathbb{Q}$: $a \cdot b = \frac{a}{b}$ - not for $b = 0$ ✗

e.g. $2 \cdot 0 \notin \mathbb{Q}$

$\mathbb{Q}^+$: $a \cdot b = \frac{a}{b}$ ✓

Group axioms

1) Associativity $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

2) There is an identity element e

$$a \cdot e = \boxed{a = e \cdot a} \quad \forall a \in G$$

$\Rightarrow$ e is unique

e.g. Addition: $e = 0$, multiplication $e = 1$

rotations of sphere: don't rotate

3) There is an inverse a^{-1}

$$\boxed{a^{-1} \cdot a = e} = a \cdot a^{-1} \quad \forall a \in G$$

eg. Addition $a = 3$, $a^{-1} = -3$

multiplication $a = 2$, $a^{-1} = \frac{1}{2}$

a^{-1} is unique

A group is Abelian if

$$a \cdot b = b \cdot a \qquad \forall\, a, b \in G$$

In this case, a and b are said to commute.

Group tables

First element in the left column, second element in the top row in the same order

1 element

$\cdot$	e
e	e

$= C_1$

(unique)

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2 elements

$\cdot$	e	x
e	e	x
x	x	$?$

Group rearrangement theorem

Every row and every column on the table contains each group element (like Sudoku)

$? = e$

Group $= C_2$ (unique)

C_2 is abelian since the table is symmetric about the main diagonal.

3 elements

$\circ$	e	γ	β
e	e	γ	β
γ	γ	$?$	
β	β		

Try $? = e$

$\circ$	e	γ	β
e	e	γ	β
γ	γ	e	
β	β	$?$	

$\uparrow$

This column has γ, e, needs β

But the bottom row already has a β. Wrong.

Must be

$\circ$	e	γ	β
e	e	γ	β
γ	γ	β	e
β	β	e	γ

$= C_3$ (unique) and abelian

check that the associative law holds.

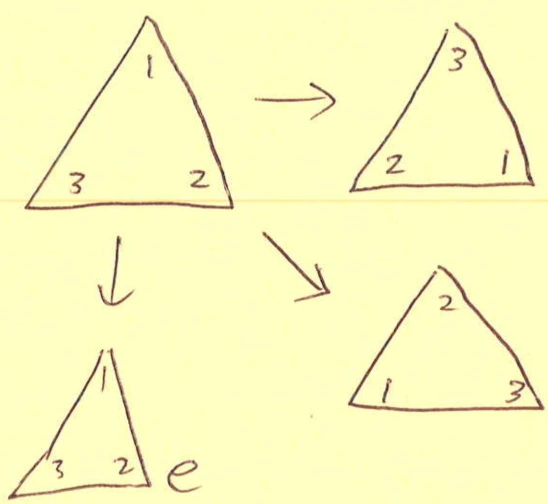
Relabeling the group elements (rearranging the rows and columns) does <u>not</u> result in a new group

$\circ$	β	γ	e
β	γ	e	β
γ	e	β	γ
e	β	γ	e

$= C_3$ same structure.

Homework: Are there any group tables for 4 elements? How many are there? Are any of them abelian?

Symmetries of a triangle that can not be flipped over (one sided):



R_{120}

R_{240}

e

binary operation is "perform one transform then the other."

$\bullet$	e	R_{120}	R_{240}
e	e	R_{120}	R_{240}
R_{120}	R_{120}	R_{240}	e
R_{240}	R_{240}	e	R_{120}

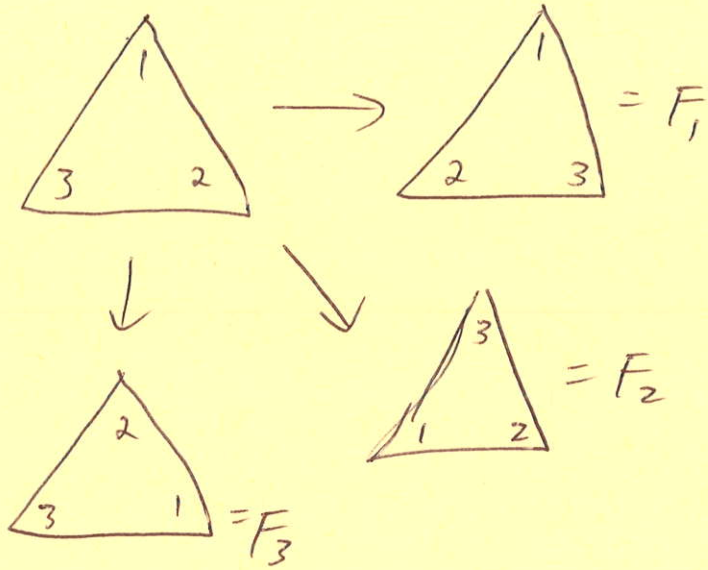
This is C_3 again.

Representation: 1-dimensional with complex numbers:

$$e = 1, \quad R_{120} = e^{\frac{i2\pi}{3}}, \quad R_{240} = e^{\frac{i4\pi}{3}}$$

See if you can find a representation with 2×2 matrices.

Now allow the triangle to be flipped over
(two sided = dihedral)



Notice the $+^{3\times3}$ block structure.
upper left is C_3.

$\bullet$	e	R_{120}	R_{240}	F_1	F_2	F_3
e	e	R_{120}	R_{240}	F_1	F_2	F_3
R_{120}	R_{120}	R_{240}	e	F_2	F_3	F_1
R_{240}	R_{240}	e	R_{120}	F_3	F_1	F_2
F_1	F_1	F_3	F_2	e	R_{240}	R_{120}
F_2	F_2	F_1	F_3	R_{120}	e	R_{240}
F_3	F_3	F_2	F_1	R_{240}	R_{120}	e

<u>Non</u> abelian

This is D_3 = dihedral group on 3 elements
also S_3 symmetric group on 3 elements (permutations).

C_3 is also A_3 the alternating group on three elements. This is the group of even permutations. S_3 is the group of all permutations.

Do the flips $\{F_1, F_2, F_3\}$ form a group?

No. There is no identity element e.

The binary operation is not closed:

e.g. $F_1 \circ F_2 = R_{240}$ which is not a flip.