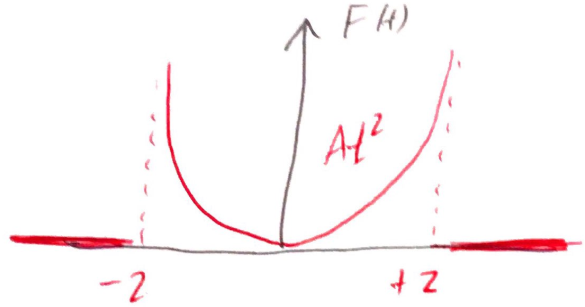
I thought it would be useful to see an example problem worked out with the Green function. Remember the Green function is the response to impulsive (delta function) forcing. Imagine a bell that you will strike once with a hammer. Before the strike, the bell does not ring. After the strike, the bell rings but the vibrations get smaller because of dissipative (damping) forces. That sound — nothing, then ringing getting softer with time — is the Green function.

There is a different Green function for every differential equation. I have only shown you the Green function for the underdamped harmonic oscillator, but the heat equation has a Green function (called the "heat kernel") and the Poisson equation from electrostatics has a Green function. They do not resemble each other.

Also note that the boundary conditions (or initial conditions) are built into the Green function. Our Green function was made for zero intial displacement $X(0)=0$ and zero initial velocity $\dot{X}(0)=0$.

Suppose the following force is applied to an underdamped harmonic oscillator:

We will need $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$



$$G(t-t') = \theta(t-t')\frac{1}{m\omega_1} e^{-\beta(t-t')} \sin[\omega_1 (t-t')]$$

The Heaviside function ensures causality - the effect should come after the cause. θ is zero when its argument is negative so $x(t)$ is affected only by forces $F(t')$ in its past, where $t-t' > 0$.

$$x(t) = \int_{t'=-\infty}^{\infty} F(t')\, G(t-t')\, dt'$$

we can replace the upper limit of integration by t because of the θ function.

Further, we can replace the lower limit by -2 since the force is zero for times $t < -2$,

$$x(t) = \int_{t'=-2}^{t} F(t')\, G(t-t')\, dt'$$

$$X(t) = \begin{cases} t < -2, & 0 \quad \leftarrow \text{no force acts on mass } m \\ -2 < t < 2, & \displaystyle\int_{t'=-2}^{t} A(t')^2 \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin[\omega_1(t-t')]\,dt' \\ & \uparrow \text{only forces from } -2 \text{ to } t \text{ act on } m. \\ 2 < t, & \displaystyle\int_{t'=2}^{+2} A(t')^2 \frac{1}{m\omega_1} e^{-\beta(t-t')} \sin[\omega_1(t-t')]\,dt' \\ & \uparrow \text{All the non-zero force acts on } m, \end{cases}$$