Eigenvalues of Hermitian matrices are real.

Hermitian: $\underline{H = H^{\dagger}} = (H^T)^*$

real symmetric matrices are Hermitian

$\underline{\underline{H}}\,\vec{v} = \lambda\,\vec{v}$

column $n \times 1$

$\begin{pmatrix} \vdots \end{pmatrix}$

Hermitian conjugate both sides

$(\underline{\underline{H}}\,\vec{v})^{\dagger} = (\lambda\,\vec{v})^{\dagger}$

$\vec{v}^{\dagger} H^{\dagger} = \lambda^* \vec{v}^{\dagger}$

row $1 \times n$ $(\cdots)$

$\vec{v}^{\dagger}\,\underline{\underline{H}}\,\vec{v} = \lambda\,\vec{v}^{\dagger}\vec{v}$

$\vec{v}^{\dagger} H^{\dagger}\vec{v} = \lambda^* \vec{v}^{\dagger}\vec{v}$

$\vec{v}^{\dagger} H\,\vec{v} = \lambda^* \vec{v}^{\dagger}\vec{v}$

subtract

$0 = (\lambda - \lambda^*)\underbrace{\vec{v}^{\dagger}\vec{v}}_{\neq 0}$

$(\cdots)\begin{pmatrix} \vdots \end{pmatrix} = (\cdot)$

$\lambda = \lambda^*$

$Re(\lambda) + Im(\lambda) = Re(\lambda) - Im(\lambda)$

$Im(\lambda) = -Im(\lambda) \Rightarrow Im(\lambda) = 0$

Eigenvectors of Hermitian matrices corresponding to different eigenvalues are orthogonal.

$\underline{\underline{H}}\,\vec{v}_1 = \lambda_1\,\vec{v}_1$

$\underline{\underline{H}}\,\vec{v}_2 = \lambda_2\,\vec{v}_2$

$\underline{\lambda_1 \neq \lambda_2}$

$\vec{v}_2^{\dagger}\,\underline{\underline{H}}^{\dagger} = \lambda_2^*\,\vec{v}_2^{\dagger}$

$\vec{v}_2^{\dagger}\,\underline{\underline{H}} = \lambda_2\,\vec{v}_2^{*}$

$\vec{v}_2^{\,\dagger} \underline{\underline{H}}\, \vec{v}_1 = \lambda_1 \vec{v}_2^{\,\dagger} \vec{v}_1 \quad \Big| \quad \vec{v}_2^{\,\dagger} \underline{\underline{H}}\, \vec{v}_1 = \lambda_2 \vec{v}_2^{\,\dagger} \vec{v}_1$

subtract

$$0 = (\lambda_1 - \lambda_2)\, \vec{v}_2^{\,\dagger} \vec{v}_1 \Rightarrow \underbrace{\vec{v}_2^{\,\dagger} \vec{v}_1}_{\text{dot product for complex vectors.}} = 0$$

What if $\lambda_1 = \lambda_2$?

Degenerate eigenvalues → eigenvectors can be chosen to be orthogonal.

Diagonalize a matrix – similarity transformation

Look for an orthogonal matrix $\underline{\underline{O}}$ such that

$\underline{\underline{O}}^T \underline{\underline{M}}\, \underline{\underline{O}}$ is diagonal $\quad \underline{\underline{O}}^{-1} \underline{\underline{M}}\, \underline{\underline{O}}$

columns of $\underline{\underline{O}}$ are eigenvectors of $\underline{\underline{M}}$

e.g. $\underline{\underline{M}} = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$ find eigenvalues.

$\det(\underline{\underline{M}} - \lambda \underline{\underline{I}}_3) = 0 \Rightarrow$ characteristic equation or eigenvalue equation

$$\det \begin{pmatrix} -\lambda & 2 & 2 \\ 2 & -\lambda & 2 \\ 2 & 2 & -\lambda \end{pmatrix} = 0 \quad = 16 + 12\lambda - \lambda^3 = (4-\lambda)(\lambda+2)^2$$

three roots: $\lambda_1 = 4 \qquad \underbrace{\lambda_2 = \lambda_3 = -2}_{\text{degenerate}}$

double root – multiplicity 2

Find eigenfns:

$\lambda_1 = 4 \qquad (\underline{\underline{M}} - \lambda_1 \underline{\underline{I}}_3)\vec{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \begin{pmatrix} v_{1x} \\ v_{1y} \\ v_{1z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

First row: $-4v_{1x} + 2v_{1y} + 2v_{1z} = 0$
2nd row: $2v_{1x} - 4v_{1y} + 2v_{1z} = 0$
2 linearly independent equations

① + 2(②) $\Rightarrow -6v_{1y} + 6v_{1z} = 0 \Rightarrow v_{1y} = v_{1z}$

Choose $v_{1y} = 1$, $v_{1z} = 1$

① $-4v_{1x} + 2 + 2 = 0 \Rightarrow v_{1x} = 1$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ normalized: $\hat{v}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2 \qquad \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} v_{2x} \\ v_{2y} \\ v_{2z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

① $2v_{2x} + 2v_{2y} + 2v_{2z} = 0$

not unique: choose $\vec{v}_2 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \rightarrow \hat{v}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$\hat{v}_2 \cdot \hat{v}_1 = 0$ ✓

$\lambda_3 = -2$: $2v_{3x} + 2v_{3y} + 2v_{3z} = 0$

$\vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \longrightarrow \hat{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$$\underline{\underline{O}} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\det \underline{\underline{M}} = \det(\underline{\underline{O}}^{+} \underline{\underline{M}}\, \underline{\underline{O}}) = 16$$

$$\mathrm{tr}\, \underline{\underline{M}} = \mathrm{tr}(\underline{\underline{O}}^{+} \underline{\underline{M}}\, \underline{\underline{O}}) = 0$$

$\underline{\underline{O}}^{+} \underline{\underline{O}} = \mathbb{I}_3 \rightarrow \underline{\underline{O}}$ is orthogonal

$$\underline{\underline{O}}^{T} \underline{\underline{M}}\, \underline{\underline{O}} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$ show this.

Spectral Decomposition of Hermitian matrices.

$$\underline{\underline{H}}\, \vec{v}_i = \lambda_i \vec{v}_i$$

Normalized

The matrix $\hat{v}_i \hat{v}_i^{*} = \underline{\underline{P}}_i$ is a projector

$$\underset{n\times 1}{\begin{pmatrix} \vdots \end{pmatrix}} \underset{1\times n}{\begin{pmatrix} \cdots \end{pmatrix}} = \underset{n\times n}{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}$$

e.g. $\hat{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\hat{x}^{+} = \begin{pmatrix} a^* & b^* & c^* \end{pmatrix}$

$$\hat{x}\hat{x}^{+} = \begin{pmatrix} aa^* & ab^* & ac^* \\ ba^* & bb^* & bc^* \\ ca^* & cb^* & cc^* \end{pmatrix}$$

Properties $\underline{\underline{P}}_i \, \underline{\underline{P}}_j = \delta_{ij} \, \underline{\underline{P}}_i$

$\underline{\underline{P}}_1 \underline{\underline{P}}_1 = \underline{\underline{P}}_1 \quad // \quad \underline{\underline{P}}_2 \underline{\underline{P}}_2 = \underline{\underline{P}}_2 \quad // \quad \underline{\underline{P}}_1 \underline{\underline{P}}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\underline{\underline{P}}_1^3 = (\underline{\underline{P}}_1 \underline{\underline{P}}_1) \underline{\underline{P}}_1 = \underline{\underline{P}}_1 \underline{\underline{P}}_1 = \underline{\underline{P}}_1 \Rightarrow \underline{\underline{P}}_i^n = \underline{\underline{P}}_i$$

Spectral Decomposition

Set of eigenvalues = "spectrum"

$$\underline{\underline{M}} = \sum_{j=1}^{n} \lambda_j \, \underline{\underline{P}}_j$$

e.g. $\underline{\underline{M}} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$ $\quad \lambda_1 = 4 \quad \hat{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = 9 \quad \hat{v}_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{\underline{P}}_1 = \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix}$ $\quad \underline{\underline{P}}_2 = \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$

$\underline{\underline{P}}_1 \underline{\underline{P}}_1 \overset{?}{=} \underline{\underline{P}}_1 \qquad \underline{\underline{P}}_2 \underline{\underline{P}}_2 = \underline{\underline{P}}_2 \qquad \underline{\underline{P}}_1 \underline{\underline{P}}_2 = \underline{\underline{0}}$

$$\sum_{j=1}^{2} \lambda_j \underline{\underline{P}}_j = \lambda_1 \underline{\underline{P}}_1 + \lambda_2 \underline{\underline{P}}_2 = 4 \begin{pmatrix} \frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{4}{5} \end{pmatrix} + 9 \begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix} \overset{?}{=} \underline{\underline{M}}$$

$$= \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix}$$

$$\underline{\underline{M}}^2 = \underline{\underline{M}}\,\underline{\underline{M}} = \left(\sum_{j=1}^{2} \lambda_j \underline{\underline{P}}_j\right)\left(\sum_{n=1}^{2} \lambda_n \underline{\underline{P}}_n\right)$$

$$= \sum_{j=1}^{2}\sum_{n=1}^{2} \lambda_j \lambda_n \underbrace{\underline{\underline{P}}_j \underline{\underline{P}}_n}_{\delta_{jn} \underline{\underline{P}}_j}$$

$$= \cancel{\sum_j} \sum_n \lambda_j \lambda_n \cancel{\delta_{jn}} \underline{\underline{P}}_j \qquad j \to n$$

$$= \sum_{n=1}^{2} \lambda_n^2 \underline{\underline{P}}_n$$

$$\underline{\underline{M}}^6 = \sum_{n=1}^{2} \lambda_n^6 \underline{\underline{P}}_n$$

Any function of a matrix

$$f(\underline{\underline{M}}) = \sum_{n=1}^{2} f(\lambda_n) \underline{\underline{P}}_n$$

easy $\sin(\underline{\underline{M}}) = \sum_{n=1}^{2} \sin(\lambda_n) \underline{\underline{P}}_n$

$$e^{\underline{\underline{M}}} = \sum_{j=1}^{2} e^{\lambda_j} \underline{\underline{P}}_n$$

hard $\sin(\underline{\underline{M}}) = \underline{\underline{M}} - \frac{\underline{\underline{M}}^3}{3!} + \frac{\underline{\underline{M}}^5}{5!} - \frac{\underline{\underline{M}}^7}{7!} + \dots$

$$\underline{\underline{M}} = \begin{pmatrix} 8 & 2 \\ 2 & 5 \end{pmatrix} \qquad \sqrt{\underline{\underline{M}}} = ?$$

$$\sqrt{\underline{\underline{M}}} = \sum_{i=1}^{2} \sqrt{\lambda_i}\, \underline{\underline{P}}_i$$

$$= \pm \frac{2}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \pm \frac{3}{5} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$