$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{+\infty} e^{i\omega t} \left[\frac{1}{\sqrt{2\pi}} \int_{t'=-\infty}^{+\infty} e^{-i\omega t'} f(t')\, dt' \right] d\omega$$

$$f(t) = \int_{t'=-\infty}^{+\infty} f(t') \underbrace{\left[\frac{1}{2\pi} \int_{\omega=-\infty}^{+\infty} e^{i\omega(t-t')}\, d\omega \right]}_{\delta(t-t')} dt'$$

Dirac delta function

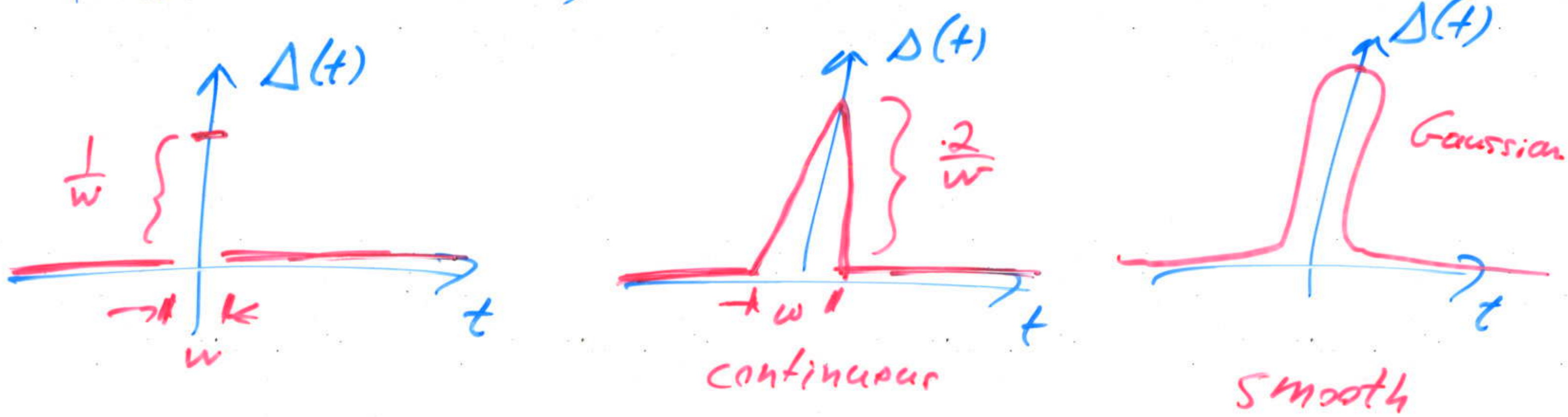
$$f(t) = \int_{t'=-\infty}^{+\infty} f(t')\, \delta(t-t')\, dt'$$

Like

$$f_i = \sum_{n=-\infty}^{+\infty} f_n \delta_{in}$$

δ_{in} - Kronecker delta

Think of $\delta(t)$ as the limits:



limit $w \to 0$
discontinuous

$\delta(x)$ was invented by Paul Dirac to describe charge distributions of point charges

$\delta(x)$ is not a function; it is a generalized function, or a distribution.

---

e.g. the proton is a ball of charge

charge density $\rho(r) = \begin{cases} \dfrac{e}{\frac{4}{3}\pi R^3} \\ \\ \end{cases}$

radius is $R \sim 1\,\text{fm}$
$= 10^{-15}\,\text{m}$

$$\iiint_V \rho_p(r)\, dV = +e$$

In contrast, an electron seems to be point. (to 10^{-19} m)

$$\rho_e(r) = -e\,\delta^3(\vec{r}) = -e\,\delta(x)\,\delta(y)\,\delta(z)$$

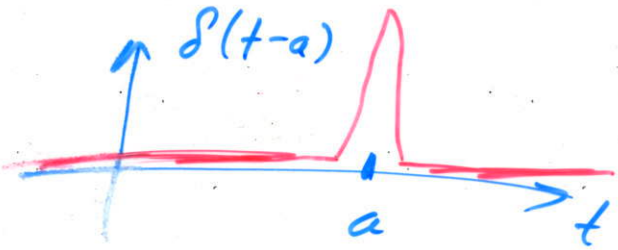
$$\iiint_V \rho_e(r)\, dV = -e$$

$$\iiint_V -e\,\delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = -e$$

$\delta(t-a)$ is zero if $t \neq a$

is infinite if $t = a$

has area under curve (integral) $= 1$



$\delta(t)$ has dimension of $\frac{1}{\text{time}}$

$$\int_{t=-\infty}^{+\infty} \delta(t)\,dt = 1$$

e.g.
$$\int_{t=-\infty}^{+\infty} f(t)\,\delta(t-3)\,dt \overset{?}{=} f(3)$$

$\delta(x)$ has dimension $\frac{1}{\text{Length}}$

$$\int_{t=-\infty}^{+\infty} f(t)\,\delta(t+3)\,dt = f(-3)$$

$$\int_{t=4}^{\infty} f(t)\,\delta(t-3)\,dt = 0$$

$$\int_{t=-\infty}^{+\infty} f(t)\,\delta(4t)\,dt \overset{?}{=}$$

change variables
$4t = x$
$4dt = dx$
$t = \infty,\ x = \infty$
$t = -\infty\ \ x = -\infty$

$$\int_{x=-\infty}^{+\infty} f\left(\frac{x}{4}\right)\delta(x)\,\frac{1}{4}\,dx = \frac{f(0)}{4}$$

Grads: $\delta(x^2)$ and $\delta[g(x)]$

Fourier transform of a delta "function."

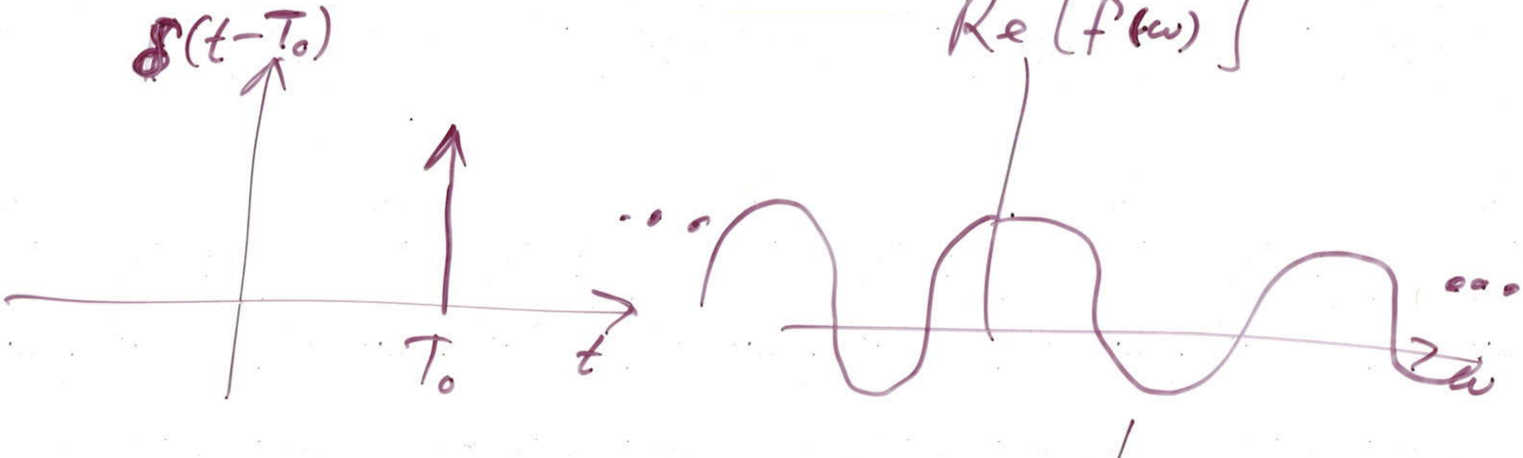
$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{+\infty} f(t)\, e^{+i\omega t}\, dt \qquad f(t) = \delta(t)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{+\infty} \delta(t)\, e^{+i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} e^{0} = \frac{1}{\sqrt{2\pi}}$$

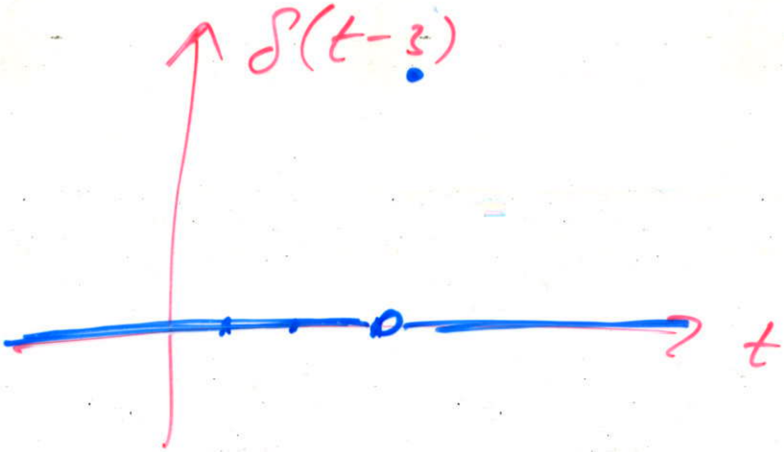
$$f(t) = \delta(t - T_0)$$

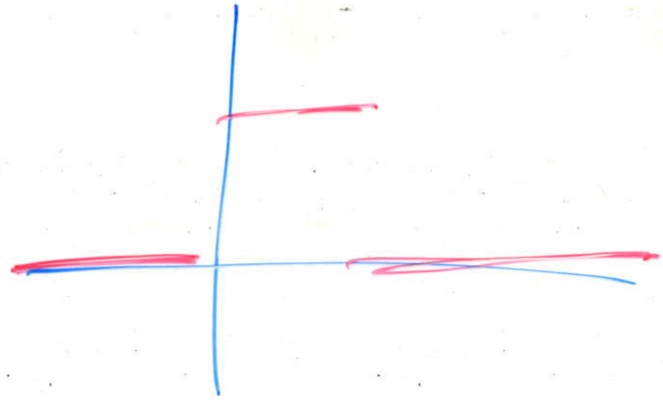
$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{+\infty} \delta(t - T_0)\, e^{i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} e^{i\omega T_0}$$

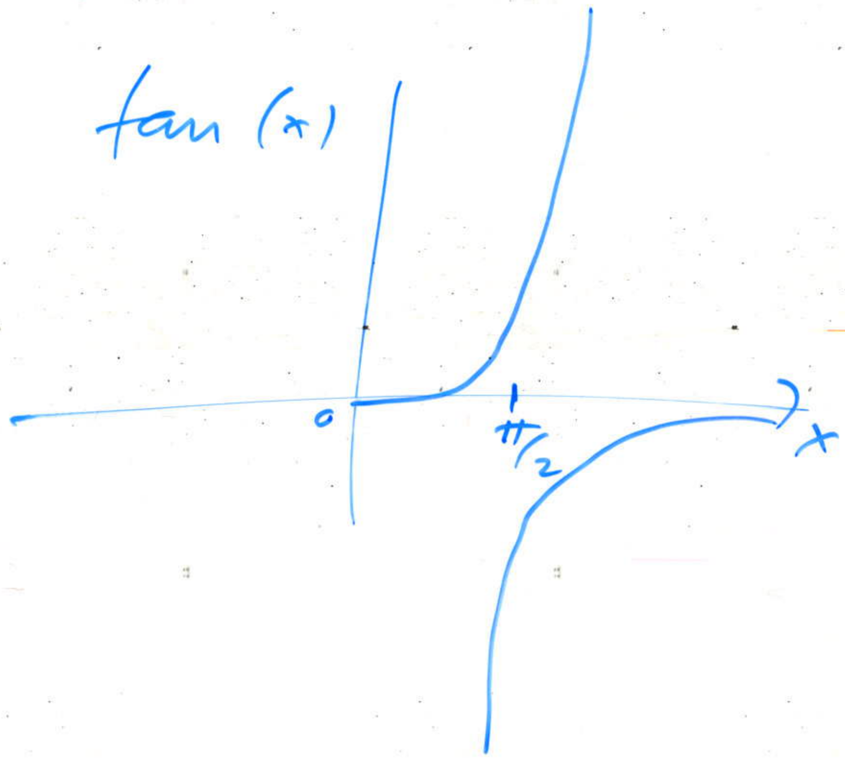
$$= \frac{1}{\sqrt{2\pi}} \left[\cos(\omega T_0) + i \sin(\omega T_0) \right]$$



$$\cos(\omega t) = \frac{1}{2}\left[e^{+i\omega t} + e^{-i\omega t} \right] \qquad \sin(\omega t) = \frac{1}{2i}\left[e^{+i\omega t} - e^{-i\omega t} \right]$$







$\delta(t)$ is not a function because ...

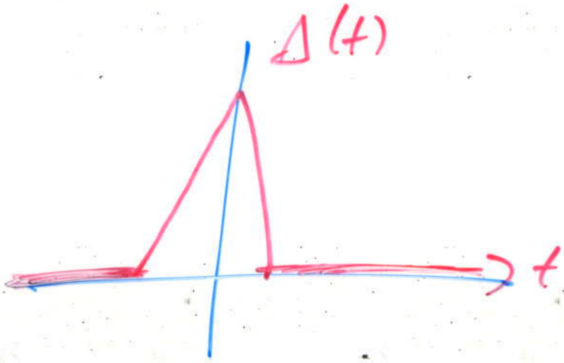
for a real function $g(t)$ $\displaystyle\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} g(t+\epsilon)\,dt = 0$

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \delta(t+\epsilon)\,dt = 1 \neq 0$$

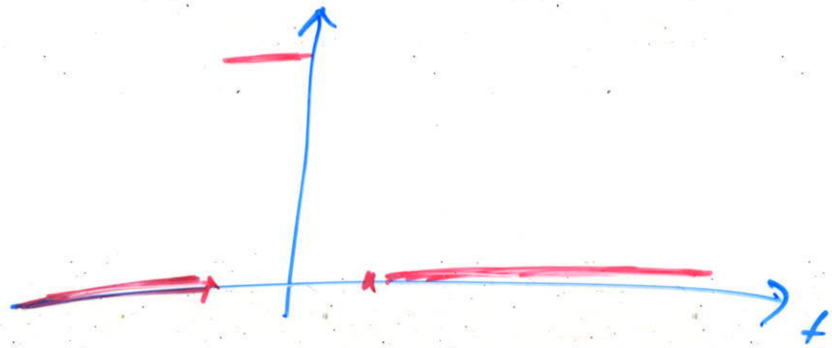
Derivative of the Dirac delta "function"

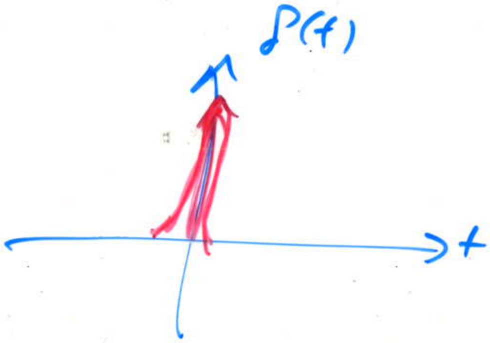
$$\delta'(t) \equiv \frac{d}{dt}\left[\delta(t)\right]$$

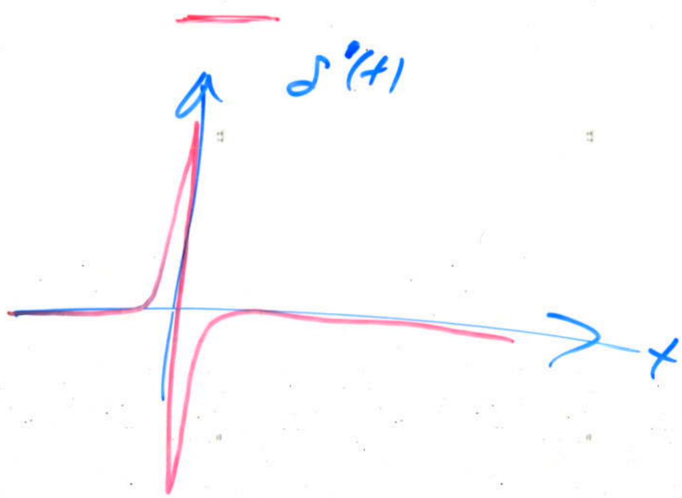
Remember



$\Delta'(t) = \frac{d\Delta}{dt}$

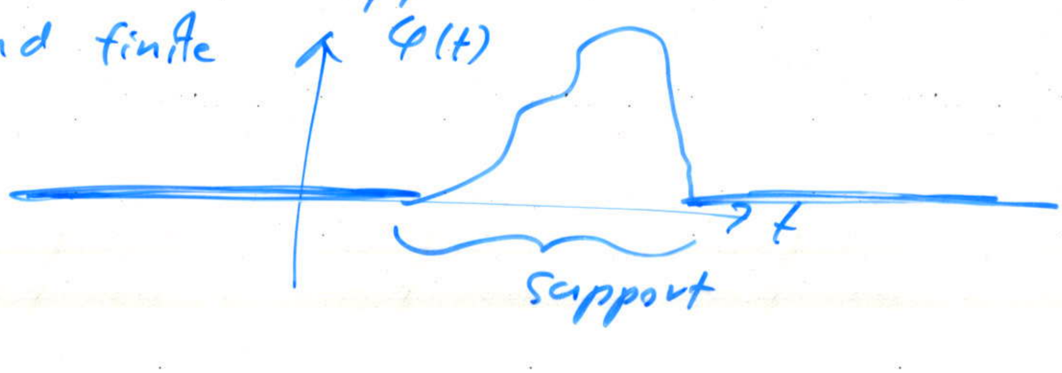






a test function $\varphi(x)$ or $\varphi(t)$ has "bounded support": zero outside some range

and finite



Generalized functions like $\delta(x)$, $\delta'(x)$, $\delta''(x)$, etc. only make sense multiplied by a test function $\varphi(x)$ and under an integral.

$$I = \int_{t=-\infty}^{+\infty} \delta'(t)\,\varphi(t)\,dt = \int_{t=-\infty}^{+\infty} \frac{d\,\delta(t)}{dt}\,\varphi(t)\,dt$$

use integration by parts

$$\int_a^b d(uv) = \int_a^b (du)\,v + \int_a^b u\,(dv)$$

$$uv\Big|_a^b = \int_a^b v\,du + \int_a^b u\,dv$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$I = \cancel{\delta(t)}^{\,0}\,\cancel{\varphi(t)}^{\,0}\Big|_{-\infty}^{+\infty} - \int_{t=-\infty}^{+\infty} \delta(t)\left(\frac{d\varphi}{dt}\right)dt$$

$$= 0 - \int_{t=-\infty}^{+\infty} \delta(t)\,\varphi'(t)\,dt = -\varphi'(0) = -\frac{d\varphi}{dt}\Big|_{t=0}$$