Equivalence of Distributions

e.g. $\sin(\alpha t)\,\delta'(t)$

Distributions <u>only</u> make sense under an integral and multiplied by a test function $\varphi(t)$.

$$I = \int_{t=-\infty}^{+\infty} \underline{\sin(\alpha t)\,\delta'(t)}\,\varphi(t)\,dt \qquad \text{integrate by parts}$$

$$I = \overbrace{\sin(\alpha t)\,\delta(t)\,\varphi(t)\Big|_{t=-\infty}^{+\infty}}^{0} - \int_{t=-\infty}^{+\infty} \delta(t)\,\frac{d}{dt}\Big[\sin(\alpha t)\,\varphi(t)\Big]\,dt$$

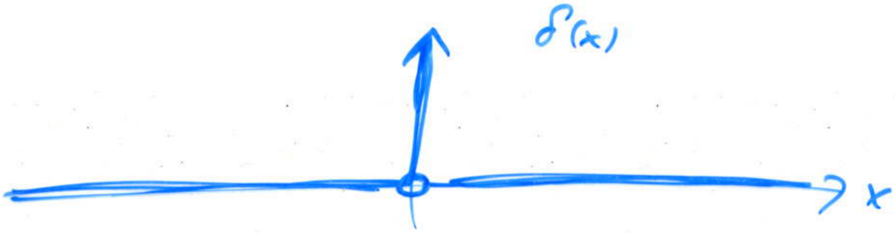
$$= -\int_{-\infty}^{+\infty} \delta(t)\Big[\alpha\cos(\alpha t)\,\varphi(t) + \sin(\alpha t)\,\varphi'(t)\Big]\,dt$$

$$= -\int_{-\infty}^{+\infty} \delta(t)\Big[\alpha \cdot 1 \cdot \varphi(0) + 0\Big]\,dt = \boxed{-\alpha\,\varphi(0)}$$

$$= \int_{t=-\infty}^{+\infty} \underline{-\alpha\,\delta(t)}\,\varphi(t)\,dt$$

$\sin(\alpha t)\,\delta'(t) \equiv -\alpha\,\delta(t)$ in the sense of distributions

Integral of Dirac delta $\delta(x)$

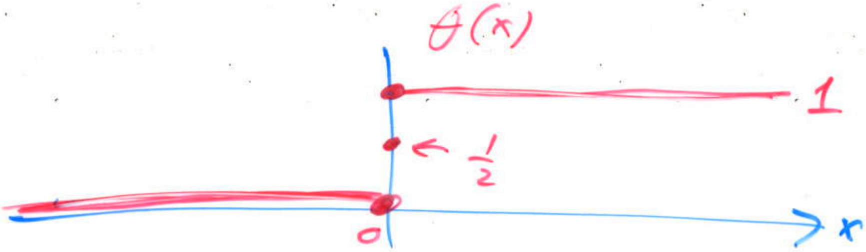


$$\theta(x) = \int_{y=-\infty}^{x} \delta(y)\, dy$$

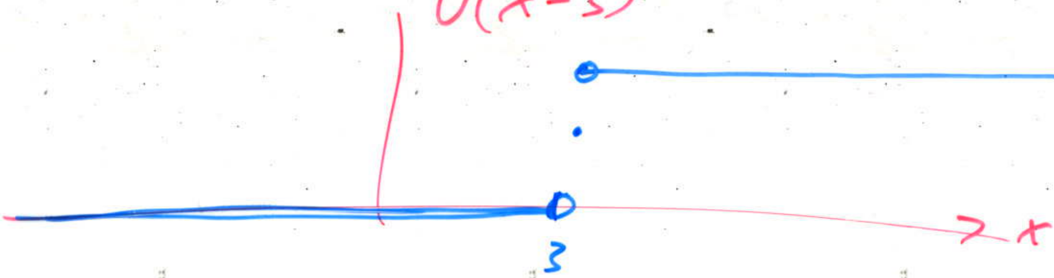
$\theta(0) = \; ? \; = H(x)$

Heaviside function

Step function



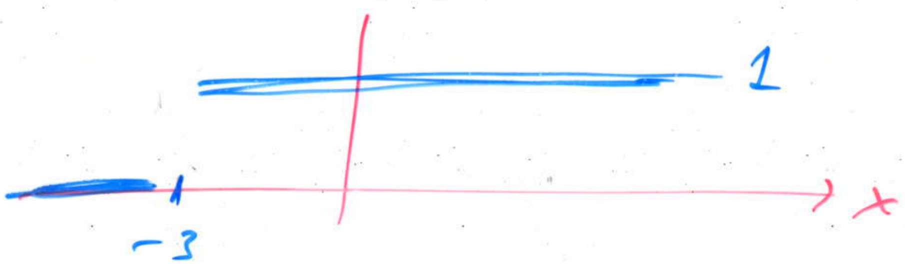
$\theta(x-3)$

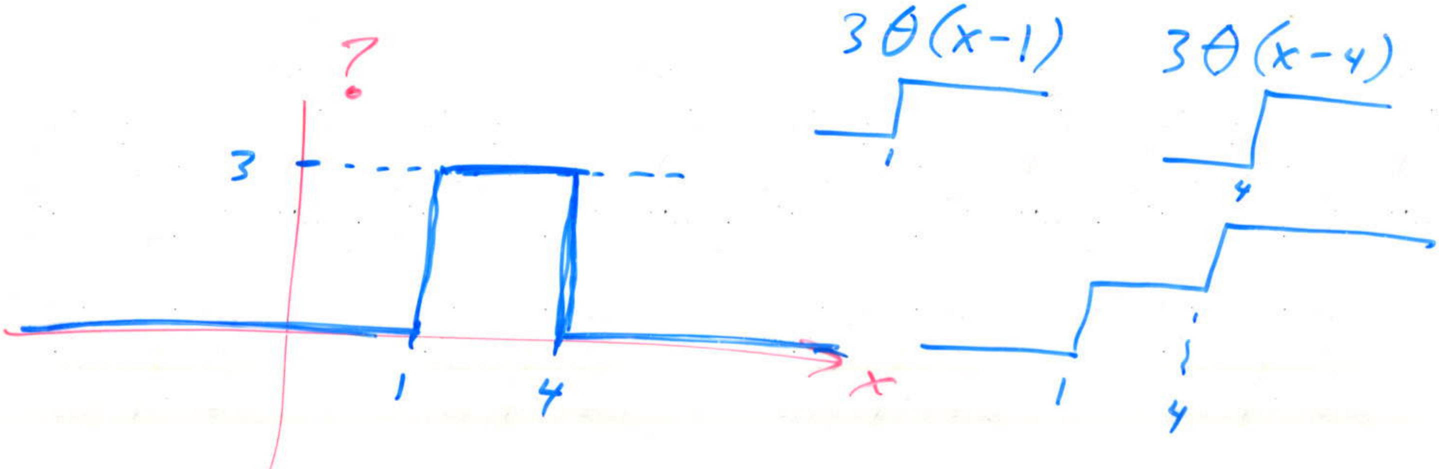


off for negative argument

on for positive argument.

$\theta(x+3)$



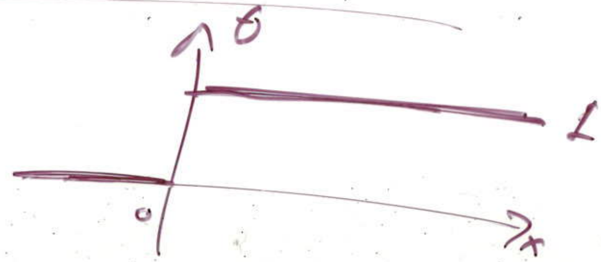


$3\theta(x-1) - 3\theta(x-4)$

$3\,\theta(x-1)\cdot\theta(x-4)$

$3\,\theta(x-1)\,\theta(4-x)$

$$\delta(x) = \frac{d\theta(x)}{dx}$$



$$\delta[f(x)] = \ ?$$

$$= \sum_{n=1}^{N} \frac{\delta[f(x_n)]}{|f'(x_n)|}$$

Call the zeroes of $f(x)$: $x_1, x_2, x_3 \ldots$

$f(x_1) = 0, \ f(x_2) = 0$

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e.g. $I = \int_{-\infty}^{+\infty} \delta(x^2-4)\,\varphi(x)\,dx \;=\; \frac{\varphi(2)}{4} + \frac{\varphi(-2)}{4}$

change variables $\quad y \equiv x^2 - 4 \qquad dy = 2x\,dx$

$$dx = \frac{1}{2}\frac{dy}{x} = \frac{dy}{2(\pm\sqrt{y+4})}$$

limits
$x = +\infty \rightarrow y = +\infty$
$x = -\infty \rightarrow y = +\infty$

y integration path