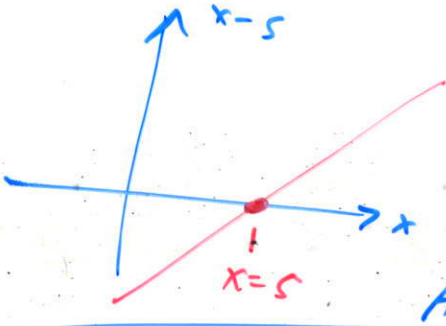
Algebraic Equations

e.g. $x-5=0$, $x^2-4=0$

find roots x_n which make the equation true

e.g. $x_1 = 5$ will solve $x-5=0$



e.g. $\left.\begin{array}{l} x_1 = +2 \\ x_2 = -2 \end{array}\right\}$ both solve $x^2-4=0$

Fundamental Theorem of Algebra

$Ax^n + Bx^{n-1} + \ldots = 0 \Longrightarrow n$ roots.

Differential Equation.

e.g. $\frac{d}{dx} f(x) = \cos(x)$

linear

function

variable

no $f(x)$, no $f'(x)$,

$\Longrightarrow$ non-homogeneous

find functions $f(x)$ which make the equation true $\forall x$

$\boxed{\text{Order}}$ - power of the highest derivative, say n'th

equation above is first order.

$\Longrightarrow$ n linearly independent solutions : $f_1(x)$, $f_2(x)$, ... $f_n(x)$

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e.g. $y'(x) - y(x) = 0$ $\Leftarrow$ homogeneous

linear

1st derivative

0th derivation

$\boxed{\text{homogeneous}}$ — there is no term in the D.E. without a derivative of the function (including the zeroeth derivative)

$\boxed{\text{linear}}$ — all derivatives of the function (including zeroeth) occur to the 1st power.

Derivatives: $f(x)$, $f'(x) = \frac{df(x)}{dx} = f^{(1)}(x)$

$$f''(x) = \frac{d^2f(x)}{dx^2} = f^{(2)}(x)$$ not f squared.

e.g.

$$\left.\begin{array}{l} y''(x) - y^2(x) = 0 \\ y'(x) + 3\sin[y(x)] = 0 \\ [y''(x)]^2 + 3x = 0 \end{array}\right\} \text{non-linear}$$

$y''(x) - 3x^2 = 0$ 2nd order, linear in y(x), non-homogeneous, ordinary differential equations.

$\boxed{\text{ordinary}}$ — one variable

$\boxed{\text{partial}}$ — more than one variable

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t) = -i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)$$

$$\Psi(\vec{r},t) = \psi(\vec{r})\, e^{-Et/\hbar}$$

differential operator

$$\underbrace{A_{(x)}\frac{d^2 f(x)}{dx^2} + B_{(x)}\frac{df(x)}{dx} + C_{(x)} f(x)}_{D[f(x)]} = 0$$

$D[f(x)]$ where $D = A\frac{d^2}{dx^2} + B\frac{d}{dx} + C$

* Coefficient $A(x)$, $B(x)$, $C(x)$ need not be constant.

The general solution $y_g(x)$ to a nonhomogeneous differential equation is the complementary solution $y_c(x)$ + the particular solution $y_p(x)$.

no arbitrary constants in here.

$$y_g(x) = y_c(x) + y_p(x)$$

$y_c(x)$ is the solution to the homogeneous equation

as many as the order of the DE.

each solution contains an arbitrary constant.

Those get fixed by boundary conditions also called initial conditions.

$y'(x) - y(x) = 7$ first order, linear in y, nonhomogeneous ordinary DE

$$y_g(x) = y_c(x) + y_p(x)$$

Find complementary solution $y_c(x)$

$$y_c'(x) - y_c(x) = 0$$

Guess $y_c(x) = 3x^2$, $y_c'(x) = 6x$ plug in

$6x - 3x^2 = 0 \quad \forall x \longrightarrow$ not a solution

Try: $y_c(x) = Ae^x$, $y_c'(x) = Ae^x$ plug in

$Ae^x - Ae^x = 0 \quad \forall x$ independent of A

A must be set using boundary conditions

Guess a particular solution: $y_p(x) = C$ = constant

$y_p'(x) = \dfrac{dC}{dx} = 0$

$y_p'(x) - y_p(x) = 7$

$0 - C = 7 \longrightarrow C = -7$

General solution $y_g(x) = y_c(x) + y_p(x)$

$= Ae^x + (-7)$

Examples of boundary conditions:

$y_g(0) = 3$ or $y_g(1) = 14$ or $y_g'(2) = 33$

$A - 7 = 3$

$A = 10$

$y_g(x) = 10e^x - 7$

e.g. $$\frac{d^2 f(t)}{dt^2} + f(t) = t^2$$ $\underline{2^{nd}}$ order, linear, nonhomogeneous, ordinary D.E.

B.C. $\left\{\begin{array}{l} f_g(0) = 2 \\ f_g'(0) = 7 \end{array}\right\}$ 2 because there will be 2 arbitrary constants in the complementary solution $f_c(t)$.

$$f_g(t) = f_c(t) + f_p(t)$$

solution to homogeneous equation

$$\ddot{f}_c(t) + f_c(t) = 0 \qquad \left\{\begin{array}{l} f_{c_1}(t) = A\sin(t) + B\cos(t) \\ f_{c_2}(t) = Ce^{it} + De^{-it} \\ f_{c_3}(t) = E\sin(t+F) \\ f_{c_4}(t) = G\cos(t+H) \end{array}\right.$$

$$\dot{f}_{c_1}(t) = A\cos(t) - B\sin(t)$$

$$\ddot{f}_{c_1}(t) = -A\sin(t) - B\cos(t) = -f_{c_1}(t)$$

$$\dot{f}_{c_2}(t) = iCe^{it} - iDe^{-it}$$

$$\ddot{f}_{c_2}(t) = i^2Ce^{it} - (-i^2)De^{-it} = -f_{c_2}(t)$$

$$\dot{f}_{c_3}(t) = E\cos(t+F), \quad \ddot{f}_{c_3}(t) = -E\sin(t+F)$$

Now the particular solution:

$$\frac{d^2 f_p(t)}{dt^2} + f_p(t) = t^2$$

feel free to overguess

Guess

$$f_p(t) = at^2 + bt + c \quad [+dt^3 + fe^t]$$

$$\dot{f}_p(t) = 2at + b$$

$$\ddot{f}_p(t) = 2a$$

now plug in

$$\Rightarrow 2a + at^2 + bt + c = t^2$$

$$\Rightarrow \underbrace{(a-1)}_{a=1}t^2 + \underbrace{b}_{b=0}t^1 + \underbrace{(2a+c)}_{c=-2a=-2}t^0 = 0$$

$\Rightarrow f_p(t) = 1t^2 + 0 - 2$

$$f_g(t) = \underbrace{A\sin(t) + B\cos(t)}_{f_c(t)} + \underbrace{1t^2 - 2}_{f_p(t)}$$

Check! then check again.

Boundary conditions → solve for A and B

$$f_g(0) = 2 = B + 0^2 - 2 \Longrightarrow B = 4$$

$$\dot{f}_g(0) = 7 = A - B(0) + 0 \Longrightarrow A = 7$$

$$\dot{f}_g(t) = A\cos(t) - B\sin(t) + 2t + 0$$