Cauchy's Theorem: (Cauchy - Goursat Theorem)
Contour integration.

If $f(z)$ is analytic (holomorphic) no infinities (poles) enclosed by the curve C,

$$\oint_C f(z)\,dz = 0$$

$dz = dx + i\,dy$

$d\vec{s} = dx\,\hat{x} + dy\,\hat{y}$

$$\oint_C (u+iv)(dx+i\,dy) = \oint_C (u\,dx - v\,dy) + i\oint (v\,dx + u\,dy)$$

$$= \oint_C (u\hat{x} - v\hat{y})\cdot d\vec{s} + i\oint_C (v\hat{x} + u\hat{y})\cdot d\vec{s}$$ line integrals

Stokes' Theorem.

$$\iint_S \vec{\nabla}\times(u\hat{x} - v\hat{y})\,\underbrace{dA}_{dx\,dy} + i\iint_S \underbrace{\vec{\nabla}\times(v\hat{x}+u\hat{y})}_{\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0}\,dA$$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & -v & 0 \end{vmatrix} = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

by C-R equations

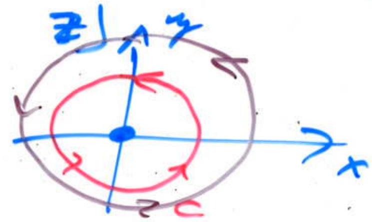
Cauchy - Goursat Theorem

$$\oint_C f(z)\,dz = 0$$

Now poles inside contour C.

$f(z) = \frac{1}{z}$ has a simple pole at $z = 0$.

e.g. $$\oint_{|z|=1} \frac{dz}{z} = I$$



parametrize $z(\theta) = 1e^{i\theta}$ on C

$$dz = i e^{i\theta} d\theta$$

$$I = \int_{\theta=0}^{2\pi} \frac{i e^{i\theta} d\theta}{e^{i\theta}} = i\int_0^{2\pi} d\theta = 2\pi i = 2\pi i \sum \text{Residue inside } C$$

Notice: same result if $C: |z| = 2$

Indefinite Integral

$$\int \frac{dz}{z} = \log(z) = \log(re^{i\theta}) = \log(r) + \log(e^{i\theta})$$

$$= \log(r) + i\theta = \log(|z|) + i\arg(z)$$

also called $ph(z)$

Definite Integral

$$I = \oint_{|z|=1} \frac{dz}{z} = \Big[\underbrace{\log(|z|)}_{=\log(1)=0} + i\arg(z)\Big]_{z=e^{0}}^{z=e^{2\pi i}} = i(2\pi - 0) = 2\pi i$$

Notice: could start $z = i e^{i\frac{\pi}{2}}$, and end at $e^{i\left(\frac{5\pi}{2}\right)}$

Cauchy's Integral formula $f(z)$ is analytic inside contour C.

$$\oint_c \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0) \leftarrow \text{Residue at } z_0$$

$\uparrow$ integrand has a simple pole at z_0

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e.g. $$I = \oint_c \frac{\cos(z)}{z^3+z}\,dz$$ Use partial fractions

$$\frac{1}{z^3+z} = \frac{1}{z(z^2+1)} = \frac{1}{z(z+i)(z-i)} = \frac{A}{z} + \frac{B}{z+i} + \frac{C}{z-i}$$

$$= \frac{A(z+i)(z-i) + Bz(z-i) + Cz(z+i)}{z(z+i)(z-i)} = \frac{1}{z^3+z}$$

$$\Rightarrow Az^2 + A + Bz^2 + Bz(-i) + Cz^2 + Czi = 1 + 0i$$

z^2: $A+B+C = 0 \Rightarrow 1+2B=0 \Rightarrow B = -\frac{1}{2}$

z^1: $-iB + iC = 0 \Rightarrow B = C$ $\Rightarrow C = -\frac{1}{2}$

z^0: $A = 1$ $A = 1$

$$\frac{1}{z^3+z} = \frac{1}{z} - \frac{1/2}{z+i} - \frac{\frac{1}{2}}{z-i}$$

Residues:

Expand $f(z)$ in a Laurent series (like a Taylor series but with negative powers of $(z-z_0)$ as well as positive), around the pole at $z=z_0$.

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_0)^k$$

Residue

$$= \dots \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

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e.g. $f(z) = z^2 \sin\left(\frac{1}{z}\right) = z^2\left(\frac{1}{z} - \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} - + \dots\right)$

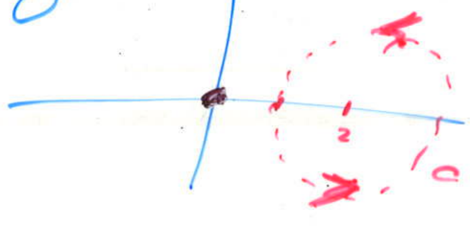
$$= z - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}\frac{1}{z^3} - + \dots$$

$\text{Res} = -\frac{1}{6}$ $\qquad = (z-0)^1 - \frac{1}{6}(z-0)^{-1} + \frac{1}{120}(z-0)^{-3}$

$$\oint_{|z|=3} z^2 \sin\left(\tfrac{1}{z}\right) dz = 2\pi i\, \text{Res}_0 = 2\pi i\left(-\tfrac{1}{6}\right) = -\frac{\pi i}{3}$$

essential singularity at $z=0$.

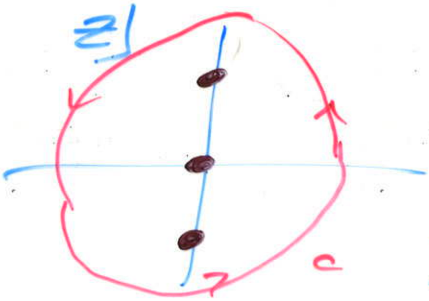
$$\oint_{|z-2|=1} z^2 \sin\left(\tfrac{1}{z}\right) dz = 0$$



$$I = \oint_c \frac{\cos(z)}{z^3+z}\,dz = \oint_c \frac{\cos(z)}{z}\,dz - \frac{1}{2}\oint_c \frac{\cos(z)}{z+i}\,dz - \frac{1}{2}\oint_c \frac{\cos(z)}{z-i}\,dz$$

$z_0 = 0$ $\qquad$ $z_0 = -i$ $\qquad$ $z_0 = +i$

$f(z) = \cos(z)$



a) $C: |z| = 2$ includes all 3 poles

$$I = 2\pi i \sum \text{Residues} = 2\pi i\left[\text{Res}_0 - \frac{1}{2}\text{Res}_{-i} - \frac{1}{2}\text{Res}_{i}\right]$$

$$= 2\pi i\left[\cos(0) - \frac{1}{2}\cos(-i) - \frac{1}{2}\cos(i)\right]$$

$$= 2\pi i\left[1 - \cosh(1)\right]$$

$$\left.\begin{aligned} e^{iz} &= \cos(z) + i\sin(z) \\ e^{-iz} &= \cos(z) - i\sin(z) \end{aligned}\right\} \quad \frac{e^{iz}+e^{-iz}}{2} = \cos(z)$$

$$\cos(i) = \frac{e^{i^2}+e^{-i^2}}{2} = \frac{e^{-1}+e^{+1}}{2} = \cosh(1)$$

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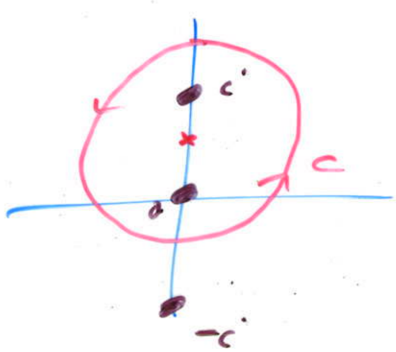
b) $C: \left|z - \frac{i}{2}\right| = 1$ only two poles inside C.



$$I = 2\pi i\left[\cos(0) - \frac{1}{2}\cos(+i)\right]$$

$$= 2\pi i\left[1 - \frac{1}{2}\cosh(1)\right]$$

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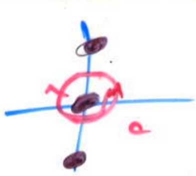
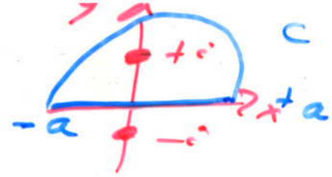
c) $C: |z| = \frac{1}{2}$



$$I = 2\pi i\left(\cos(0)\right) = 2\pi i$$

$$I = \int_{x=-\infty}^{+\infty} \frac{dx}{(x^2+1)^2} = ?$$

$$f(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$$



Two poles of order 2; one pole at $z=-i$, one pole at $z=+i$

$$\oint_C f(z)\,dz = \int_{x=-a}^{+a} \frac{dx}{(x^2+1)^2} + \int_{\text{arc}} f(z)\,dz$$

$$\int_{x=-a}^{+a} \frac{dx}{(x^2+1)^2} = \oint_C f(z)\,dz - \cancel{\int_{\text{arc}} f(z)\,dz} \to 0$$

~~Taylor~~ Laurent expansion of $\frac{1}{(z^2+1)^2}$ around $z=+i$

$$\underbrace{-\frac{1}{4}}_{a_{-2}} \frac{1}{(z-i)^2} \underbrace{- \frac{i}{4}}_{a_{-1}\ \text{Res}} \frac{1}{z-i} + \underbrace{\frac{3}{16}}_{a_0} + \underbrace{\frac{i}{8}}_{a_1}(z-i)^1 + \ldots \quad a_2 \ldots$$

$$\text{Res} = -\frac{i}{4}$$

$$\oint_C f(z)\,dz = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$$

$\int_{\text{arc}} f(z)\,dz$ parametrize $z = a e^{i\theta}$, $dz = a i e^{i\theta}\,d\theta$

$$\| \quad \int_{\theta=0}^{\pi} \frac{a i e^{i\theta}\,d\theta}{(a^2 e^{2i\theta}+1)^2} \sim \frac{1}{a^3} \longrightarrow 0 \text{ as } a \to \infty$$

Limit as $a \to \infty$

$$\int_{x=-\infty}^{+\infty} \frac{dx}{(x^2+1)^2} = I = \frac{\pi}{2}$$

Fibonacci Sequence : 1 1 2 3 5 8 13 21 34 55 89

$u_1 = 1$ $\quad p = 1 = q$

$u_2 = 1$

Recursion Relation: $u_{n+2} = p\, u_{n+1} + q\, u_n$

Ansatz (Guess) : $u_n = k z^n$

$$k z^{n+2} = 1\, k z^{n+1} + 1\, k z^n \qquad \div\, k z^n$$

$$\Rightarrow z^2 = z + 1 \qquad z_\pm = \left\{ \frac{1+\sqrt{5}}{2} , \frac{1-\sqrt{5}}{2} \right\}$$

In general: $u_n = k_+ z_+ + k z_-$

solve for coefficients k_+ and k_- using the initial condition $u_1 = 1$, $u_2 = 1$

$$\left.\begin{aligned} u_1 &= 1 = k_+\left(\frac{1+\sqrt{5}}{2}\right) + k_-\left(\frac{1-\sqrt{5}}{2}\right) \\ u_2 &= 1 = k_+\left(\frac{1+\sqrt{5}}{2}\right)^2 + k_-\left(\frac{1-\sqrt{5}}{2}\right)^2 \end{aligned}\right\} \quad k_+ = \frac{1}{\sqrt{5}} \quad k_- = -\frac{1}{\sqrt{5}}$$

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$