Matrices

$a x = b$ linear (in x) non-homogeneous equation in one variable (x)

$x = \frac{b}{a}$ if $a \neq 0$

$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$ (linear (in x's) non-homogeneous equation in n variables.

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &= b_m \end{aligned}\right\}$$ system of m linear equations in n variables.

$m > n$: system is overconstrained (over determined) in general, there is no solution $\{x_1, x_2 \dots x_n\}$

$m < n$: underdetermined $\Rightarrow$ infinitely many solutions.

$m = n$: interesting, either 0 or 1 solution.

Matrix notation: $\underline{\underline{A}} \vec{x} = \vec{b}$ ← $m \times 1$ matrices (column vectors)

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix}$$

↑ $m \times n$ matrix

$1 \times n$ matrix = row vector $\vec{x}^T$

Index notation: $\sum_{i=1}^{m} a_{ji} x_i = b_j$

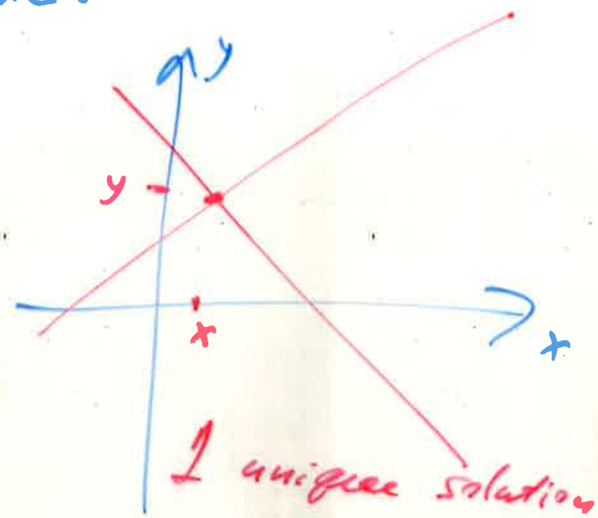
i – dummy index (summed over)

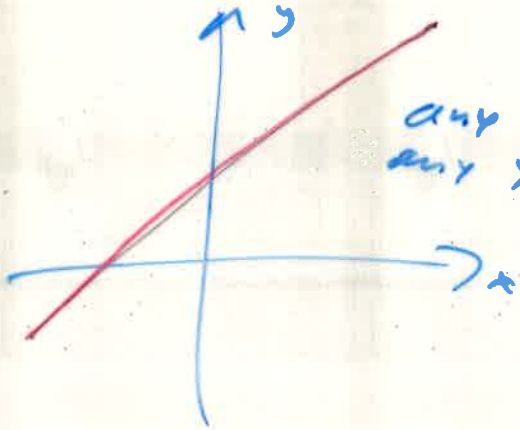
j – free index

Geometry. Look 2x2 matrices

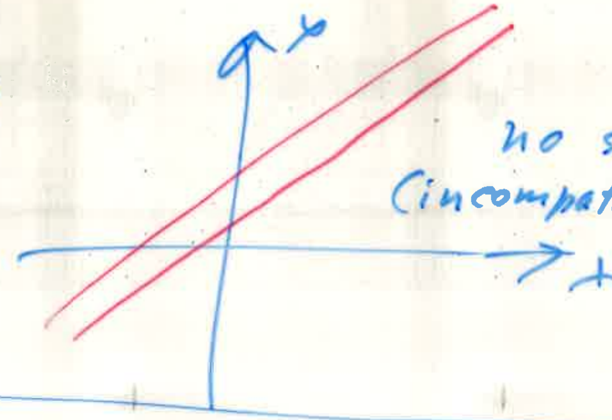
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\left.\begin{aligned} a_{11}x + a_{12}y &= c \\ a_{21}x + a_{22}y &= d \end{aligned}\right\} \text{lines}$$





any x
any y work



no solution
(incompatible)

Multiply Matrices

$$\underset{m \times n}{\underline{A}} \; \underset{n \times p}{\underline{C}} = \underset{m \times p}{\underline{D}}$$

$$\sum_{k=1}^{n} \overset{\text{number}}{a_{ik}} \, \overset{\text{number}}{C_{kj}} = \overset{\text{number}}{d_{ij}}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{pmatrix} \leftarrow \underline{D}$$

numbers (scalars, 1×1 matrices) commute under multiplication $2 \cdot 3 = 3 \cdot 2 = 6$

In general, matrices do not commute.

$$\underline{\underline{A}}\,\underline{\underline{C}} \neq \underline{\underline{C}}\,\underline{\underline{A}} \qquad \text{e.g. } A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$\underline{\underline{A}}\,\underline{\underline{C}} = \begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix} \neq \underline{\underline{C}}\,\underline{\underline{A}} \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$

Matrices do commute under addition

$$\underline{\underline{A}} + \underline{\underline{C}} = \underline{\underline{C}} + \underline{\underline{A}}$$

$$a_{ij} + c_{ij} = c_{ij} + a_{ij}$$

Scalar Multiplication of a matrix

$$s\,\underline{\underline{A}} = \underline{\underline{C}} \qquad s\,a_{ij} = c_{ij}$$

row column

$$2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

Transpose $\underline{\underline{A}}^T = \underline{\underline{C}} \qquad (a_{ij})^T = a_{ji} = c_{ij}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

A matrix is <u>symmetric</u> if $\underline{\underline{A}} = \underline{\underline{A}}^T$

e.g. $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is symmetric   3 independent comp. $\frac{n(n+1)}{2}$

A matrix is <u>antisymmetric</u> if $\underline{\underline{A}} = -\underline{\underline{A}}^T$

e.g. $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ $a_{ij} = -a_{ji}$

$a_{11} = -a_{11}$

1 independent comp.

any matrix $\underline{\underline{B}} = \underset{\uparrow \text{sym.}}{\underline{\underline{S}}} + \underset{\uparrow \text{antisymmetric}}{\underline{\underline{A}}}$

$(\underline{\underline{A}}^T)^T = \underline{\underline{A}}$ from now on, all matrices are square $n \times n$.

Matrix Addition and Multiplication are associative

$(\underline{\underline{A}} + \underline{\underline{C}}) + \underline{\underline{D}} = \underline{\underline{A}} + (\underline{\underline{C}} + \underline{\underline{D}})$

$(\underline{\underline{A}}\,\underline{\underline{C}})\underline{\underline{D}} = \underline{\underline{A}}(\underline{\underline{C}}\,\underline{\underline{D}})$

i,j free $\sum_{p=1}^{n}\left(\sum_{k=1}^{n} a_{ik} c_{kp}\right) d_{pj} \stackrel{?}{=} \sum_{k=1}^{n} a_{ik}\left(\sum_{p=1}^{n} c_{kp} d_{pj}\right)$

Zero matrix $\begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$

$$\underline{\underline{S}} = \frac{\underline{\underline{B}} + \underline{\underline{B}}^T}{2}$$

$$S^T = \frac{B^T + B^{T^T}}{2} = \frac{B^T + B}{2} = S$$

$$\underline{\underline{A}} = \frac{\underline{\underline{B}} - \underline{\underline{B}}^T}{2}$$

Identity matrix $(n \times n)$ $\quad \begin{pmatrix} 1 & 0 & & 0 \\ 0 & 1 & 0 & 0 \\ & & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 1's on main diagonal

$\underline{\underline{I}}_n$ order $\qquad (\underline{\underline{I}}_n)_{ij} = \delta_{ij} \qquad \underline{\underline{I}}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Matrix Multiplication distributes over addition

$$\underline{\underline{A}}(\underline{\underline{C}} + \underline{\underline{D}}) = \underline{\underline{A}}\,\underline{\underline{C}} + \underline{\underline{A}}\,\underline{\underline{D}}$$

Matrix exponents $\qquad p \in$ integers

$$\underline{\underline{A}}^p = \underbrace{\underline{\underline{A}}\,\underline{\underline{A}} \cdots \underline{\underline{A}}}_{p \text{ times}} \qquad \underline{\underline{A}}^0 \equiv \underline{\underline{I}}$$

$$e^{\underline{\underline{A}}} = \underline{\underline{I}} + \underline{\underline{A}} + \frac{\underline{\underline{A}}\,\underline{\underline{A}}}{2!} + \frac{\underline{\underline{A}}^3}{3!} + \cdots$$

$$\underline{\underline{A}}^p \underline{\underline{A}}^q = \underline{\underline{A}}^{p+q} = \underline{\underline{A}}^{q+p} = \underline{\underline{A}}^q \underline{\underline{A}}^p$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^p = \underbrace{\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{A}}\,\underline{\underline{B}} \cdots \underline{\underline{A}}\,\underline{\underline{B}}}_{p \text{ times}} \neq \underline{\underline{A}}^p \underline{\underline{B}}^p$$

Trace: $\quad tr(\underline{\underline{A}}) \equiv \sum_{i=1}^{n} a_{ii}$ $\quad$ sum diagonal entries

$tr\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 5 \qquad tr(\underline{\underline{I}}_n) = n$

$$(\underline{\underline{A}} + \underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T \leftarrow$$

$$\left[(\underline{\underline{A}}\,\underline{\underline{B}})^T\right]_{ij} = (\underline{\underline{A}}\,\underline{\underline{B}})_{ji}$$

$$= \sum_{k=1}^{n} a_{jk}\, b_{ki} = \sum_{k=1}^{n} a^T_{kj}\, b^T_{ik} = \sum_{k=1}^{n} b^T_{ik}\, a^T_{kj}$$

$$= \left[\underline{\underline{B}}^T \underline{\underline{A}}^T\right]_{ij}$$

Inverse: $\underline{\underline{A}}^{-1}$

$$\underline{\underline{A}}^{-1}\underline{\underline{A}}^{+1} = \underline{\underline{I}} \qquad \underline{\underline{A}}^{+1}\underline{\underline{A}}^{-1} = \underline{\underline{A}}^{0} = \underline{\underline{I}}$$

$$\underline{\underline{A}}^{-2} = (\underline{\underline{A}}^{-1})(\underline{\underline{A}}^{-1})$$

Not every matrix has an inverse

$\underline{\underline{A}}$ must be square $n \times n$ to have an inverse.

Some $n \times n$ matrices do <u>not</u> have an inverse.