Chapter 5

Differential Forms

Differential forms are, without a doubt, one of the most beautiful inventions in all of mathematics! Their elegance and simplicity are without bound. Nevertheless, like anything good, it takes a long time before they become natural and second nature to the student. Whether you realize it or not, you have been using differential forms for a long time now (gradient).

Before diving into forms, I should mention two quick things. First of all, it is likely that many of you will never need to know any of this in your life. If that is the case, then you have no reason to read this chapter (unless you're curious). I have included it anyway, since writing these notes has helped me to understand some of the more subtle details of the mathematics, and I find forms to be very useful. Differential forms are extremely useful when, for example, you wish to discuss electromagnetism, or any gauge field for that matter, in curved space (i.e.: with gravity). This is why I thought they would do well in this review.

The second point I want to make is that no matter how much you like or dislike forms, remember that they are nothing more than notation. If mathematicians had never discovered differential forms, we would not be at a serious loss; however, many calculations would become long and tedious. The idea behind this beautiful notation is that terms cancel early. If we had never heard of differential forms, one would have to carry out terms that would ultimately vanish, making the calculations nasty, at best! But let us not forget that there is nothing truly "fundamental" about forms – they are a convienience, nothing more.

That being said, let's jump in...

5.1 The Mathematician's Definition of Vector

One thing you have to be very careful of in mathematics is definitions. We have to construct a definition that both allows for all the cases that we know should exist, and also forbids any pathological examples. This is a big challenge in general mathematical research. We have been talking a lot about "vectors", where we have been using the physicist's definition involving transformation properties. But this is unfortunately not the only mathematical definition of a vector. Mathematicians have another definition that stems from the subject of linear algebra. For the moment, we are going to completely forget about the physical definition (no more transformations) and look on vectors from the mathematician's point of view. To this end we must make a new definition; the key definition of linear algebra:

Definition: A **Vector Space** (sometimes called a **Linear Space**), denoted by $\mathcal{V}$, is a set of objects, called **vectors**, and a field, called **scalars**, and two operations called vector addition and scalar multiplication that has the following properties:

1. $(\mathcal{V},+)$ is an abelian group.
2. $\mathcal{V}$ is closed, associative, and distributive under scalar multiplication, and there is a scalar identity element $(1\vec{v} = \vec{v})$.

In our case, the field of scalars will always be $\mathbb{R}$, but in general it could be something else, like $\mathbb{C}$ or some other field from abstract algebra.

Notice that this definition of vectors has absolutely nothing to do with the transformation properties that we are used to dealing with. Nevertheless, you can prove to yourself very easily that the set of n-dimensional real vectors ($\mathbb{R}^n$) is a vector space. But so are much more abstract quantities. For example, the set of all n-differentiable, real-valued functions ($C^n(\mathbb{R})$) is also a vector space, where the "vectors" are the functions themselves. Another example of a vector space is the set of all eigenfunctions in quantum mechanics, where the eigenfunctions are the vectors. This is why it's called "abstract" algebra!

Sticking to abstract linear algebra for a minute, there is a very important fact about vector spaces. Recall that a **linear transformation** is a function that operates on $\mathcal{V}$ and preserves addition and scalar multiplication:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

Then let us consider the set of all linear transformations that go from $\mathcal{V}$ to $\mathbb{R}$:

$$\Lambda^1(\mathcal{V}) = \{\phi | \ \ \phi : \mathcal{V} \to \mathbb{R}; \ \ \phi \text{ linear}\} \tag{5.1.1}$$

I'll explain the notation in a moment. This set is very intriguing - every element in this set has the property that it acts on any vector in the original vector space and returns a number. It is, in general, true that for any vector space, this special set exists. It is also true that this space also forms a vector space itself. It is so important in linear algebra, it is given a name. We call it the **dual space of** $\mathcal{V}$, and denote it as $\mathcal{V}^*$. The vectors (functions) in the space are called **covectors**, or **1-forms** (hense the 1 superscript).

Armed with this knowledge we can go back and redefine what we mean by "tensor": consider the set of all multilinear (that is, linear in each of its arguments) transformations that send k vectors and l covectors to a number:

$$T : \underbrace{\mathcal{V} \times \cdots \times \mathcal{V}}_{\text{k times}} \times \underbrace{\mathcal{V}^* \times \cdots \times \mathcal{V})^*}_{\text{l times}} \to \mathbb{R}$$

The set of all such transformations is called T_k^l and the elements of this set are called **tensors of rank** $\begin{pmatrix} l \\ k \end{pmatrix}$. Notice immediately that $T_1^0 = \mathcal{V}$, $T_0^1 = \mathcal{V}^*$. Also notice that this definition of tensor has nothing to do with transformations, as we defined them in Chapter 1. For example, acording to this definition, the Christoffel symbols of differential geometry *are* $(1,2)$ tensors. So you must be sure you know which definition people are using.

That's enough abstraction for one day. Let us consider a concrete example. Consider the vector space $\mathbb{R}^n$ and the set of *multilinear* transformations that take k vectors to a number:

$$\phi : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{\text{k times}} \to \mathbb{R} \tag{5.1.2}$$

with the additional property that they are "alternating", i.e.:

$$\phi(\ldots, \vec{x}, \vec{y}, \ldots) = -\phi(\ldots, \vec{y}, \vec{x}, \ldots) \tag{5.1.3}$$

When ϕ takes k vectors in $\mathbb{R}^n$, then ϕ is called a **k-form**. The set of all k-forms is denoted $\Lambda^k(\mathbb{R}^n)$. Note that this set is also a vector space. The dual space is an example of this, hence the notation.

In linear algebra, we know that every vector space has a basis, and the dimension of the vector space is equal to the number of elements in a basis. What is the standard basis for $\Lambda^k(\mathbb{R}^n)$? Let us start by considering the dual space to $\mathbb{R}^n$ ($k = 1$). We know the standard basis for $\mathbb{R}^n$; let's denote it by $\{\vec{e}_1, ..., \vec{e}_n\}$. Now let us *define* a 1-form dx^i with the property:

$$dx^i(\vec{e_j}) \equiv \left\langle dx^i, \vec{e_j} \right\rangle = \delta^i_j \tag{5.1.4}$$

Then the **standard basis of $\Lambda^k(\mathbb{R}^n)$** is the set of all k-forms:

$$dx^i \wedge dx^j \wedge \cdots dx^l; \quad i \neq j \neq \cdots \neq l; \quad i, j, \cdots, l \leq n \tag{5.1.5}$$

where each "wedge product" has k "dx"s[1].

Before going any further, perhaps it would be wise to go over this and make sure you understand it. Can you write down the standard basis for $\Lambda^2(\mathbb{R}^3)$? How about $\Lambda^2(\mathbb{R}^4)$? Here are the answers:

$$\begin{aligned}
\Lambda^2(\mathbb{R}^3) \quad &: \quad \{dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3\} \Rightarrow D = 3 \\
\Lambda^2(\mathbb{R}^4) \quad &: \quad \{dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^1 \wedge dx^4, dx^2 \wedge dx^3, dx^2 \wedge dx^4, dx^3 \wedge dx^4\} \Rightarrow D = 6
\end{aligned}$$

By looking at these examples, you should be able to figure out what the dimension of $\Lambda^k(\mathbb{R}^n)$ is, in general ($k < n$):

$$\dim \Lambda^k(\mathbb{R}^n) = \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{5.1.6}$$

Notice that if $k > n$ the space is trivial – it only has the zero element.

[1]Wedge products are discussed below, but for now all you need to know of them is that they are anti-symmetric products, so $A \wedge B = -B \wedge A$ for A, B 1−forms.

5.2 Form Operations

5.2.1 Wedge Product

In order to define the standard basis for $\Lambda^k(\mathbb{R}^n)$, we needed to introduce a new operation, called the **wedge product** ($\wedge$). Let's look more carefully at this new form of multiplication. As a "pseudo-definition", consider two 1-forms $\omega, \phi \in \Lambda^1(\mathbb{R}^n)$; then let us define an element in $\Lambda^2(\mathbb{R}^n)$ by the following:

$$\omega \wedge \phi(\vec{v}, \vec{w}) = \omega(\vec{v})\phi(\vec{w}) - \omega(\vec{w})\phi(\vec{v}) \tag{5.2.7}$$

Notice that there are two immediate corollaries:

$$\begin{aligned} \omega \wedge \phi &= -\phi \wedge \omega && (5.2.8) \\ \omega \wedge \omega &= 0 \qquad \forall \omega \in \Lambda^k(\mathbb{R}^n) && (5.2.9) \end{aligned}$$

So the wedge product defines a noncommutative product of forms. Notice that this is different from the ordinary tensor product:

$$\omega \otimes \phi(\vec{v}, \vec{w}) \equiv \omega(\vec{v})\phi(\vec{w}) \tag{5.2.10}$$

Looking at this as an antisymmetric product on forms, you might be reminded of the cross product. This is exactly right: the cross-product of two vectors is the same thing as a wedge product of two 1-forms. We will prove this explicitly soon.

Now that we have talked about the specific case of 1-forms, we can generalize to the wedge product of any forms:

Definition: Let $\omega \in \Lambda^k(\mathbb{R}^n), \phi \in \Lambda^l(\mathbb{R}^n)$. Then we define $\omega \wedge \phi \in \Lambda^{k+l}(\mathbb{R}^n)$ as the sum of all the antisymmetric combinations of wedge products. This product is

1. Distributive: $(dx^1 + dx^2) \wedge dx^3 = dx^1 \wedge dx^2 + dx^1 \wedge dx^3$
2. Associative: $(dx^1 \wedge dx^2) \wedge dx^3 = dx^1 \wedge (dx^2 \wedge dx^3) = dx^1 \wedge dx^2 \wedge dx^3$
3. Skew-commutative: $\omega \wedge \phi = (-1)^{kl} \phi \wedge \omega$

Notice that for $k + l > n$ the space is trivial, so $\phi \wedge \omega = 0$.

Before going any further, let's do some examples. Consider $\omega, \phi \in \Lambda^1(\mathbb{R}^3)$, i.e.: 1-forms in 3-space. Writing it out explicitly, we have:

$$\begin{aligned} \omega &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 \\ \phi &= b_1 dx^1 + b_2 dx^2 + b_3 dx^3 \\ \omega \wedge \phi &= (a_1 dx^1 + a_2 dx^2 + a_3 dx^3) \wedge (b_1 dx^1 + b_2 dx^2 + b_3 dx^3) \\ &= (a_1 b_2 - a_2 b_1) dx^1 \wedge dx^2 + (a_1 b_3 - a_3 b_1) dx^1 \wedge dx^3 + (a_2 b_3 - a_3 b_2) dx^2 \wedge dx^3 \end{aligned}$$

Notice how I've been careful to keep track of minus signs. Also notice that this looks very similar to a cross product, as stated earlier. What about taking the wedge product of two 1-forms in $\mathbb{R}^4$:

$$\begin{aligned}
\xi &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 + a_4 dx^4 \\
\chi &= b_1 dx^1 + b_2 dx^2 + b_3 dx^3 + b_4 dx^4 \\
\xi \wedge \chi &= (a_1 b_2 - a_2 b_1) dx^1 \wedge dx^2 + (a_1 b_3 - a_3 b_1) dx^1 \wedge dx^3 + (a_1 b_4 - a_4 b_1) dx^1 \wedge dx^4 \\
&\quad + (a_2 b_3 - a_3 b_2) dx^2 \wedge dx^3 + (a_2 b_4 - a_4 b_2) dx^2 \wedge dx^4 + (a_3 b_4 - a_4 b_3) dx^3 \wedge dx^4
\end{aligned}$$

Note that this does not look like a "cross product"; it is a six-dimensional object in $\mathbb{R}^4$. However, if you took another wedge product with another 1-form, you would get a 3-form in $\mathbb{R}^4$ (which is four-dimensional), and that would give you something like a four-dimensional cross product. In general, taking the wedge product of $(n-1)$ 1-forms in $\mathbb{R}^n$ will give you something analogous to a cross-product. We'll get back to this later.

As a final example, what if you took the wedge product of a 1-form and a 2-form in $\mathbb{R}^3$:

$$\alpha = c_1 dx^1 + c_2 dx^2 + c_3 dx^3 \tag{5.2.11}$$

$$\beta = L_{12} dx^1 \wedge dx^2 + L_{13} dx^1 \wedge dx^3 + L_{23} dx^2 \wedge dx^3 \tag{5.2.12}$$

$$\alpha \wedge \beta = (c_1 L_{23} - c_2 L_{13} + c_3 L_{12}) dx^1 \wedge dx^2 \wedge dx^3 \tag{5.2.13}$$

Notice that if we identify $\beta = \omega \wedge \phi$ from above, then we have shown that the triple wedge product of 1-forms in $\mathbb{R}^3$ is just:

$$\omega \wedge \phi \wedge \alpha = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} dx^1 \wedge dx^2 \wedge dx^3 \tag{5.2.14}$$

In fact, the wedge product of n 1-forms in $\mathbb{R}^n$ is always such a determinant. This is a very important fact that will come up again and again.

Next, we can ask how a general wedge product of forms acts on vectors? The answer is that the form farthest to the left acts on the vector, and then you must permute the form so that all the 1-forms act on the vector. As an example, consider a 2-form acting on a vector:

$$dx \wedge dy(\vec{v}) = dx(\vec{v})dy - dy(\vec{v})dx = v_x dy - v_y dx$$

The minus sign comes from flipping the dx and the dy.

Before moving on, let us conclude this section with a definition and a theorem:

Definition: A **simple k-form** is a "monomial" k-form; that is, there is no addition.

Theorem: All k-forms can be written as a linear combination of simple k-forms.

Proof: The proof of this theorem is straightforward: a simple k-form is just a basis element, possibly multiplied by a number. Then since we are dealing with a vector space, any element of $\Lambda^k(\mathbb{R}^n)$ can be written as a linear combination of these basis elements. **QED.**

5.2.2 Tilde

The tilde operator is the operator that translates us from vectors to forms, and vice versa. In general, it is an operation: (rank-k tensors)$\mapsto$(k-forms). Let's see some examples:

$$\begin{aligned}
\vec{v} &= v^1\vec{e}_1 + v^2\vec{e}_2 + v^3\vec{e}_3 \in \mathbb{R}^3 \\
\tilde{v} &= v_1 dx^1 + v_2 dx^2 + v_3 dx^3 \in \Lambda^1(\mathbb{R}^3)
\end{aligned}$$

Notice that the tilde changed subscript indices to superscript. This is important when we have to keep track of what is covariant and contravariant.

Let's do another less trivial example:

$$\begin{aligned}
L &= \begin{pmatrix} 0 & L_{12} & L_{13} \\ -L_{12} & 0 & L_{23} \\ -L_{13} & -L_{23} & 0 \end{pmatrix} \in \mathbb{R}^3 \quad \text{(Rank-2)} \\
\tilde{L} &= \left(L_{12}dx^1 \wedge dx^2 + L_{13}dx^1 \wedge dx^3 + L_{23}dx^2 \wedge dx^3\right) \in \Lambda^2(\mathbb{R}^3) \\
&= \frac{1}{2} L_{ij} dx^i \wedge dx^j
\end{aligned}$$

where in the last line I'm using Einstein notation for the indices. Notice how much simpler it is to write a 2-form as opposed to writing a rank-2 tensor. Also note that both L_{ij} and L_{ji} are taken care of, including the minus sign, by using wedge products as opposed to using tensor products - this is why we need the factor of $\frac{1}{2}$ in the last line. We are beginning to see why form notation can be so useful!

Incidentally, notice that I am only considering antisymmetric tensors when I talk about forms. This is because the form notation has antisymmetry built into it. For symmetric tensors, the concept of a form is not very useful.

Finally, let me state a pretty obvious result:

Theorem: $\tilde{\tilde{v}} = v$.

Proof: Left to the reader.

5.2.3 Hodge Star

The next tool we introduce for differential forms is the **Hodge Star** ($*$). The Hodge Star converts forms to their so-called **dual form**:

$$* : \Lambda^k(\mathbb{R}^n) \to \Lambda^{n-k}(\mathbb{R}^n) \tag{5.2.15}$$

As for many other things in linear algebra, it is sufficient to consider how the Hodge star acts on the basis elements. Let's look at some examples.

In $\mathbb{R}^2$:

$$\begin{aligned}
*dx^1 &= dx^2 \\
*dx^2 &= -dx^1
\end{aligned}$$

In $\mathbb{R}^3$:

$$\begin{array}{rclrcl} *dx^1 & = & dx^2 \wedge dx^3 & *(dx^1 \wedge dx^2) & = & dx^3 \\ *dx^2 & = & dx^3 \wedge dx^1 & *(dx^2 \wedge dx^3) & = & dx^1 \\ *dx^3 & = & dx^1 \wedge dx^2 & *(dx^3 \wedge dx^1) & = & dx^2 \end{array}$$

In $\mathbb{R}^4$:

$$\begin{array}{rclrclrcl} *dx^1 & = & +dx^2 \wedge dx^3 \wedge dx^4 & *(dx^1 \wedge dx^2) & = & +dx^3 \wedge dx^4 & *(dx^1 \wedge dx^2 \wedge dx^3) & = & +dx^4 \\ *dx^2 & = & -dx^3 \wedge dx^4 \wedge dx^1 & *(dx^1 \wedge dx^3) & = & -dx^2 \wedge dx^4 & *(dx^1 \wedge dx^2 \wedge dx^4) & = & -dx^3 \\ *dx^3 & = & +dx^4 \wedge dx^1 \wedge dx^2 & *(dx^1 \wedge dx^4) & = & +dx^2 \wedge dx^3 & *(dx^1 \wedge dx^3 \wedge dx^4) & = & +dx^2 \\ *dx^4 & = & -dx^1 \wedge dx^2 \wedge dx^3 & *(dx^2 \wedge dx^3) & = & +dx^1 \wedge dx^4 & *(dx^2 \wedge dx^3 \wedge dx^4) & = & -dx^1 \\ & & & *(dx^2 \wedge dx^4) & = & -dx^1 \wedge dx^3 & & & \\ & & & *(dx^3 \wedge dx^4) & = & +dx^1 \wedge dx^2 & & & \end{array}$$

We can see a pattern here if we remember how the Levi-Civita simbol works. For any form in $\Lambda^k(\mathbb{R}^n)$:

$$\Phi = \phi_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k} \Rightarrow$$

$$\boxed{*\Phi = \frac{1}{(n-k)!} \epsilon_{\nu_1 \dots \nu_n} \phi^{\nu_1 \dots \nu_k} dx^{\nu_{k+1}} \wedge \cdots \wedge dx^{\nu_n}} \tag{5.2.16}$$

or

$$\boxed{*\phi_{\mu_1 \dots \mu_{n-k}} = \frac{1}{(n-k)!} \epsilon_{\nu_1 \dots \nu_k \mu_1 \dots \mu_{n-k}} \phi^{\nu_1 \dots \nu_k}} \tag{5.2.17}$$

From the above, we can now present a theorem:

Theorem: For any k-form $\omega \in \Lambda^k(\mathbb{R}^n)$, $** \omega = (-1)^{n-1} \omega$.

The proof follows from using Equation (5.2.16) and the properties of the Levi-Civita tensor. Notice that the double-Hodge star does *not* depend on k at all!

Note that in all cases, we get the general result:

$$dx^i \wedge *dx^i = \bigwedge_{j=1}^{n} dx^j \equiv d^n x \in \Lambda^n(\mathbb{R}^n) \tag{5.2.18}$$

$\Lambda^n(\mathbb{R}^n)$ is one-dimensional; it's single basis element is the wedge product of all the dx^j. This product is called the **volume form of** $\mathbb{R}^n$. With this in mind, we can reinterpret Equation (5.2.14). Given three vectors in $\mathbb{R}^3$, take their corresponding forms with the tilde and then take the wedge product of the result. The answer you get is the determinant of the matrix formed by these three vectors, multiplied by the volume form of $\mathbb{R}^3$. What is this determinant? It is simply the Jacobian of the transformation from the standard basis to the basis of the three vectors you chose! In other words, differential forms automatically give us the vector calculus result:

$$d^n\vec{x}' = \mathcal{J} d^n\vec{x} \tag{5.2.19}$$

5.2.4 Evaluating k-Forms

We have done a lot of defining, but not too much calculating. You might be asking at this point: OK, so we have all of this machinery- now what?

The appearance of determinants should give you a clue. Whenever we need to evaluate a k-form, we construct a matrix according to the prescription the form gives us. Finally, we take the determinant of the matrix.

To construct the matrix, we simply chose whatever elements of our vectors get picked out by the simple form.

Examples: evalutate the following forms:

1. $$dx \wedge dy \left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right]$$

2. $$dx \wedge dz \left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right]$$

3. $$dx \wedge dy \wedge dz \left[\begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} d \\ e \\ f \end{pmatrix}, \begin{pmatrix} g \\ h \\ i \end{pmatrix} \right]$$

Answers:

1. Construct the matrix by picking out the "x" and "y" components of the vectors, and take the determinant:
$$\det \begin{pmatrix} a & d \\ b & e \end{pmatrix}$$

2. This is similar, but now pick out the "x" and "z" components:
$$\det \begin{pmatrix} a & d \\ c & f \end{pmatrix}$$

3. For this case, it is simply what we had before:
$$\det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Hopefully you get the hang of it by now.

5.2.5 Generalized Cross Product

Eariler, I mentioned that we can define a generalized cross product in $\mathbb{R}^n$ by wedging together $(n-1)$ 1-forms. Let's see this explicitly:

In $\mathbb{R}^n$, take $n-1$ vectors, and tilde them so they are now 1-forms: $\{\tilde{v}_1, \ldots, \tilde{v}_{n-1}\}$. Now wedge these forms together to give you an $(n-1)$-form, and take the Hodge star of the product to give us a 1-form. Finally tilde the whole thing to give us an n-vector, and we have defined a general product of vectors:

$$\boxed{\vec{v}_1 \times \cdots \times \vec{v}_{n-1} \equiv [\widetilde{*(\tilde{v}_1 \wedge \cdots \wedge \tilde{v}_{n-1})}] \in \mathbb{R}^n} \tag{5.2.20}$$

5.3 Exterior Calculus

A **k-form field** is a k-form whose coefficients depend on the coordinates. This is exactly analogous to a vector field in vector calculus. So, if we have a vector field in $\mathbb{R}^3$:

$$\begin{aligned}\vec{v}(\vec{x}) &= v^1(\vec{x})\vec{e}_1 + v^2(\vec{x})\vec{e}_2 + v^3(\vec{x})\vec{e}_3 \\ \tilde{v}(\vec{x}) &= v_1(\vec{x})dx^1 + v_2(\vec{x})dx^2 + v_3(\vec{x})dx^3\end{aligned}$$

5.3.1 Exterior Derivative

Now that we have a notion of "function", let's see what we can do in the way of calculus. We can define an operator

$$d : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k+1}(\mathbb{R}^n)$$

with the all-important property:

$$d^2 = dd \equiv 0 \tag{5.3.21}$$

By construction we will let d act on forms in the following way:

$$d(a(x)dx) = da(x) \wedge dx \tag{5.3.22}$$

Let's look at some examples.

$$f(\vec{x}) \in \Lambda^0(\mathbb{R}^3) \to df(\vec{x}) = \left(\frac{\partial f}{\partial x^1}\right) dx^1 + \left(\frac{\partial f}{\partial x^2}\right) dx^2 + \left(\frac{\partial f}{\partial x^3}\right) dx^3 \in \Lambda^1(\mathbb{R}^3)$$

This is just the gradient of a function! How about the curl?

$$
\begin{aligned}
\omega &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 \to \\
d\omega &= \left[\left(\frac{\partial a_1}{\partial x^1}\right) dx^1 + \left(\frac{\partial a_1}{\partial x^2}\right) dx^2 + \left(\frac{\partial a_1}{\partial x^3}\right) dx^3\right] \wedge dx^1 \\
&\quad + \left[\left(\frac{\partial a_2}{\partial x^1}\right) dx^1 + \left(\frac{\partial a_2}{\partial x^2}\right) dx^2 + \left(\frac{\partial a_2}{\partial x^3}\right) dx^3\right] \wedge dx^2 \\
&\quad + \left[\left(\frac{\partial a_3}{\partial x^1}\right) dx^1 + \left(\frac{\partial a_3}{\partial x^2}\right) dx^2 + \left(\frac{\partial a_3}{\partial x^3}\right) dx^3\right] \wedge dx^3 \\
&= \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right) dx^1 \wedge dx^2 + \left(\frac{\partial a_3}{\partial x^1} - \frac{\partial a_1}{\partial x^3}\right) dx^1 \wedge dx^3 + \left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}\right) dx^2 \wedge dx^3
\end{aligned}
$$

Notice that terms like $\left(\frac{\partial a_1}{\partial x^1}\right) dx^1 \wedge dx^1$ vanish immediately because of the wedge product. Again we see how the beauty of differential forms notation pays off!

5.3.2 Formulas from Vector Calculus

At this point we can immediately write down some expressions. Some of them I have shown. Can you prove the rest of them? In all cases, f is any (differentiable) function, and $\vec{v}$ is a vector field.

$$
\begin{aligned}
\widetilde{df} &= \nabla f && (5.3.23) \\
\widetilde{*d\tilde{v}} &= \nabla \times \vec{v} \quad (\text{in } \mathbb{R}^3) && (5.3.24) \\
d\tilde{v} &= \nabla \cdot \vec{v} && (5.3.25) \\
*d*df &= \nabla^2 f && (5.3.26)
\end{aligned}
$$

Notice that even though $dd = 0$, $d * d \neq 0$ in general!

5.3.3 Orthonormal Coordinates

Differential forms allow us to do calculations without ever refering to a coordinate system. However, sooner or later we will want to get our answers back into a coordinate frame. This could be tricky.

The key point to remember is that each of these dx^i is an *orthonormal* coordinate[2]. Therefore we must make sure to translate them into orthonormal coordinates in our coordinate frame; otherwise, the Hodge star will not work.

Let's consider a simple example by looking at the two-dimensional Laplacian. From above, we know that $\nabla^2 = *d*d$. Cartesian coorindates ($x^1 = x, x^2 = y$) are orthonormal, so we can easily plug into our formula:

[2] That means that $< dx^i, \vec{e}_j >= \delta^i_j$ and $< \vec{e}_i, \vec{e}_j >= \delta_{ij}$.

$$
\begin{aligned}
df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\
*df &= \frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx \\
d*df &= \frac{\partial^2 f}{\partial x^2}dx \wedge dy - \frac{\partial^2 f}{\partial y^2}dy \wedge dx \\
&= \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] dx \wedge dy \\
*d*df &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}
\end{aligned}
$$

All is well with the world. But what if I wanted to do my work in polar coordinates? Then my coordinate system is $(x^1 = r, x^2 = \theta)$, but the *orthonormal* polar coordinate basis vectors are $\vec{\mathbf{e}}_1 = \hat{\mathbf{r}}, \vec{\mathbf{e}}_2 = \frac{\hat{\theta}}{r}$, and the dual basis is $(dr, rd\theta)$ (see previous footnote). So:

$$
\begin{aligned}
df &= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta\left(\frac{r}{r}\right) = \frac{\partial f}{\partial r}dr + \frac{1}{r}\frac{\partial f}{\partial \theta}(rd\theta) \\
*df &= \frac{\partial f}{\partial r}(rd\theta) - \frac{1}{r}\frac{\partial f}{\partial \theta}dr \\
d*df &= \left\{\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) dr \wedge d\theta - \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) d\theta \wedge dr\right\}\left(\frac{r}{r}\right) \\
&= \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}\right] dr \wedge (rd\theta) \\
*d*df &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \nabla^2 f \qquad \text{Wow!}
\end{aligned}
$$

where I have introduced $(\frac{r}{r})$ in the first and third line in order to keep the coordinates orthonormal before using the Hodge star. So, as long as we are careful to use orthonormal coordinates when using the Hodge star, our formulas work for *any* coordinate system! This is a vital reason why differential forms are so useful when working with general relativity, where you want to prove things about the physics independent of your reference frame or coordinate basis.

5.4 Integration

We have discussed how to differentiate k-forms; how about integrating them? First I will discuss how to formally evaluate forms, and then I will present a formula for integrating them. With this definition in mind, we will be able to derive Stokes' Theorem- possibly the most important theorem differential forms will provide us with.

5.4.1 Evaluating k-form Fields

First, a definition:

Definition: An **Oriented k-Parallelogram**, denoted by $\pm P^0_{\vec{x}}\{\vec{v}_i\}$, $(i = 1, \ldots, k)$, is a k-parallelogram spanned by the k n-vectors $\{\vec{v}_i\}$ with basepoint $\vec{x}$. $P^0_{\vec{x}}\{\vec{v}_i\}$ is antisymmetric under $\vec{v}_i$ exchange.

Oriented k-Parallelograms allow you to think geometrically about evaluating k-forms, but as far as paperwork goes, they are just a way to keep track of what vectors are what. We use them explicitly to evaluate k-form fields at a point. Let's see how that works:

Let's evaluate the 2-form field $\phi = \cos(xz)dx \wedge dy \in \Lambda^2(\mathbb{R}^3)$. We'll evaluate it on the oriented 2-parallelogram spanned by the 3-vectors $\vec{v}_1 = (1, 0, 1)$ and $\vec{v}_2 = (2, 2, 3)$. We'll evaluate it at two points: $(1, 2, \pi)$ and $(\frac{1}{2}, 2, \pi)$:

1. $$\phi \left[P^0_{\begin{pmatrix} 1 \\ 2 \\ \pi \end{pmatrix}} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\} \right] = \cos(1 \cdot \pi) \det \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = -2$$

2. $$\phi \left[P^0_{\begin{pmatrix} \frac{1}{2} \\ 2 \\ \pi \end{pmatrix}} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\} \right] = \cos(\frac{1}{2} \cdot \pi) \det \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = 0$$

5.4.2 Integrating k-form Fields

Now that we have a formal way to evaluate k-form fields, we can talk about integrating them. We *define* what the integral of a k-form field is:

Definition: Let $\phi \in \Lambda^k(\mathbb{R}^n)$ be a k-form field, $A \subset \mathbb{R}^k$ be a pavable set, and $\gamma : A \to \mathbb{R}^n$ be a (vector-valued) differentiable map. Then we define the integral of ϕ over $\gamma(A)$ as:

$$\boxed{\int_{\gamma(A)} \phi \equiv \int_A \phi \left[P^0_{\gamma(\vec{u})} \left\{ \left.\frac{\partial \vec{\gamma}}{\partial x^1}\right|_{\vec{u}}, \left.\frac{\partial \vec{\gamma}}{\partial x^2}\right|_{\vec{u}}, \ldots, \left.\frac{\partial \vec{\gamma}}{\partial x^k}\right|_{\vec{u}} \right\} \right] d^k\vec{u}} \tag{5.4.27}$$

where $d^k\vec{u}$ is the k-volume form.

Like in vector calculus, we can define the integral formally in terms of Riemann sums - yuck! Let me just say that this is an equivalent definition; the more curious students can go prove it.

Let's do two examples to get the hang of it:

1. Consider $\phi \in \Lambda^1(\mathbb{R}^2)$ and integrate over the map:

$$\gamma(u) = \begin{pmatrix} R\cos u \\ R\sin u \end{pmatrix}$$

over the region $A = [0, \alpha]$, $(\alpha > 0)$. Let $\phi = xdy - ydx$:

$$\begin{aligned}
\int_{\gamma(A)} \phi &= \int_{[0,\alpha]} (xdy - ydx)\left[P^0_{\begin{pmatrix} R\cos u \\ R\sin u \end{pmatrix}} \begin{pmatrix} -R\sin u \\ R\cos u \end{pmatrix}\right] du \\
&= \int_0^\alpha du[(R\cos u)(R\cos u) - (R\sin u)(-R\sin u)] \\
&= \int_0^\alpha du R^2[\cos^2 u + \sin^2 u] \\
&= \alpha R^2
\end{aligned}$$

2. Consider $\phi = dx \wedge dy + ydx \wedge dz \in \Lambda^2(\mathbb{R}^3)$, and the map:

$$\gamma\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} s+t \\ s^2 \\ t^2 \end{pmatrix}$$

over the region $C = \{\begin{pmatrix} s \\ t \end{pmatrix} | 0 \leq s \leq 1, 0 \leq t \leq 1\}$:

$$\begin{aligned}
\int_{\gamma(C)} \phi &= \int_0^1 \int_0^1 (dx \wedge dy + ydx \wedge dz)\left[P^0_{\begin{pmatrix} s+t \\ s^2 \\ t^2 \end{pmatrix}} \left\{\begin{pmatrix} 1 \\ 2s \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2t \end{pmatrix}\right\}\right] dsdt \\
&= \int_0^1 \int_0^1 \left[\det\begin{pmatrix} 1 & 1 \\ 2s & 0 \end{pmatrix} + s^2 \det\begin{pmatrix} 1 & 1 \\ 0 & 2t \end{pmatrix}\right] dsdt \\
&= \int_0^1 \int_0^1 (-2s + 2s^2 t) dsdt \\
&= \int_0^1 ds(-2s + s^2) = -\frac{2}{3}
\end{aligned}$$

5.4.3 Stokes' Theorem

Now we can move on to present one of if not the most important theorems that differential forms has to offer- Stokes' Theorem. I will present it correctly; do not be overly concerned with all the hypotheses; it suffices that the area you are integrating over has to be appropriately "nice".

Theorem: Let X be a compact, piece-with-boundary of a (k+1)-dimensional oriented manifold M$\subset \mathbb{R}^n$. Give the boundary of X (denoted by ∂X) the proper orientation, and consider a k-form field $\phi \in \Lambda^k(\mathbb{R}^n)$ defined on a neighborhood of X. Then:

$$\boxed{\int_{\partial X} \phi = \int_X d\phi} \tag{5.4.28}$$

What is this theorem saying? My old calculus professor used to call it the "Jumping-d theorem", since the "*d*" jumps from the manifold to the form. In words, this theorem says that the integral of a form over the boundary of a sufficiently nice manifold is the same thing as the integral of the derivative of the form over the whole mainfold itself.

You have used this theorem many times before. Let's rewrite it in more familiar notation, for the case of $\mathbb{R}^3$:

k	$d\phi$	X	$\int_X d\phi = \int_{\partial X} \phi$	Theorem Name
0	$\nabla f \cdot d\vec{x}$	Path from $\vec{a}$ to $\vec{b}$	$\int_{\vec{a}}^{\vec{b}} \nabla f \cdot d\vec{x} = f(\vec{b}) - f(\vec{a})$	FTOC
1	$(\nabla \times \vec{f}) \cdot d\vec{S}$	Surface (Ω)	$\int_\Omega (\nabla \times \vec{f}) \cdot d\vec{S} = \oint_{\partial\Omega} \vec{f} \cdot d\vec{x}$	Stokes Theorem
2	$(\nabla \cdot \vec{f}) d^3x$	Volume (V)	$\int_V (\nabla \cdot \vec{f}) d3x = \oint_{\partial V} \vec{f} \cdot d\vec{S}$	Gauss Theorem

Here, I am using vector notation (even though technically I am supposed to be working with forms) and for the case of $\mathbb{R}^3$, I've taken advantage of the following notations:

$$\begin{aligned} d\vec{x} &= (dx, dy, dz) \\ d\vec{S} &= (dy \wedge dz, dz \wedge dx, dx \wedge dy) \\ d^3x &= dx \wedge dy \wedge dz \end{aligned}$$

As you can see, all of the theorems of vector calculus in three dimensions are reproduced as specific cases of this generalized Stokes Theorem. However, in the forms notation, we are not limited to three dimensions!

Before leaving differential forms behind, I should mention one important point that I have gone out of my way to avoid: the issue of orientation. For the sake of this introduction, let me just say that all the manifolds we consider must be "orientable with acceptable boundary". What does this mean? It means that the manifold must have a sense of "up" (no Möbius strips). "Acceptable boundary" basically means that we can, in a consistent and smooth way, "straighten" the boundary (no fractals). These are technical issues that I have purposely left out, but they are important if you want to be consistent.

5.5 Forms and Electrodynamics

As a finale for differential forms, I thought it would be nice to summarize briefly how one uses forms in theories such as E&M. Recall that in covariant electrodynamics, we have an antisymmetric, rank-2 4-tensor known as the "Field Strength" tensor:

$$\mathbf{F} \equiv F_{\mu\nu} dx^\mu \otimes dx^\nu \tag{5.5.29}$$

As we have seen, it is always possible to use 2-forms instead of (antisymmetric) tensors, and we can rewrite the above tensor as a differential form in $\Lambda^2(M_4)$[3]:

$$F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \in \Lambda^2(M_4) \tag{5.5.30}$$

Notice that the antisymmetry of $F_{\mu\nu}$ is immediate in form notation. Also, this is a 2-form in four dimensions, which means that its dual is also a 2-form. Now using Equation (5.5.30), I can write the Lorentz Force Law for a particle with charge e moving with a velocity vector $\vec{u}$ as:

$$\dot{\tilde{p}} = eF(\vec{u}) \in \Lambda^1(M_4) \tag{5.5.31}$$

Let's consider an example. Suppose we have a magnetic field in the $\hat{\mathbf{x}}$ direction, so $F = B_x dy \wedge dz$. Then the force on the particle is:

$$\begin{aligned}
\dot{\tilde{p}} = eF(\vec{u}) &= eB_x \langle dy \wedge dz, \vec{u} \rangle \\
&= eB_x [dy \langle dz, \vec{u} \rangle - dz \langle dy, \vec{u} \rangle] \\
&= eB_x [u_z dy - u_y dz] \Rightarrow \dot{\vec{p}} = eB_x (u_z \hat{\mathbf{y}} - u_y \hat{\mathbf{z}})
\end{aligned}$$

This is exactly what you would have gotten if you had used your vector-form of Lorentz's Law, with cross products.

Another vector that is important in E&M is the 4-current (J). We will again think of J as a 1-form in M_4; therefore, it's dual is a 3-form.

Armed with all we need, we can write down Maxwell's Equations:

$$dF = 0 \tag{5.5.32}$$

$$d * F = 4\pi * J \tag{5.5.33}$$

To interpret these equations, one can think of F $(*F)$ as an object that represents "tubes of force" flowing through space-time.

We can take advantage of Stokes Theorem (just like in regular vector calculus) to put these equations in integral form:

[3] M_4 is Minkowski space-time. In general, I can go to any space I want; for now, I'm sticking to flat space (i.e.: ignoring gravity). But notice that this analysis is totally general, and gravity can be easily included at this stage.

$$\int_\Sigma dF = \oint_{\partial\Sigma} F = 0 \tag{5.5.34}$$
$$\int_\Sigma d * F = \oint_{\partial\Sigma} *F = 4\pi(\text{charge}) \tag{5.5.35}$$

The first of these equations says that the total flux of F through a closed region of space-time is zero; the second equation says that the total flux of $*F$ through a closed region of space-time is proportional to the amount of charge in that region. Notice that this description never mentions coordinate systems: once again, differential forms has allowed us to describe physics and geometry without ever referring to a coordinate system! Notice the similarity of this equation with the Gauss-Bonnet Theorem – in gauge theories, you may think of the field strength as a curvature to some space!

The final step in studying electrodynamics is to notice that Equation (5.5.32) suggests something. Recall that if a differential form was the exterior derivative of some other form, then it's exterior derivative was necessarely zero, via Equation (5.3.21). Is the converse true? Namely, if you have a form whose exterior derivative is zero, can it be written as the exterior derivative of some other form?

Your first impulse might be to say "yes": surely if the form is an exterior derivative, your condition is satisfied. But it turns out that the question is a little more subtle than that. For I can give you a form which is not an exterior derivative of another form, and yet still has vanishing exterior derivative! This is a famous case of "necessity" versus "sufficiency" in mathematics. That a form is an exterior derivative is sufficient for its exterior derivative to vanish, but not necessary.

The key point is that this property depends not on the differential forms themselves, but on the *global properties of the space* they live in! It turns out that M_4 does indeed have the property of necessity; specifically, it is simply connected. Therefore it is safe to assume a la Equation (5.5.32) that we can write:

$$F = dA \tag{5.5.36}$$

for some 1-form A. So indeed, electrodynamics in flat space can be described by a 4-vector potential. But do be careful not to jump to any conclusions before you know the properties of the universe you are describing!

Finally, we can consider what happens if we let $A \to A + d\lambda$, where λ is any (reasonably well-behaved) function. Then by just plugging into Equation (5.5.36), with the help of Equation (5.3.21), we see that F remains the same. Hense, the 4-vector potential is only defined up to a 4-gradient; this is exactly what gauge invariance tells us should happen!

This gives a very powerful geometric intuition for our field-strength, and therefore for electricity and magnetism. You can do similar analyses for any gauge field, and by adjusting the manifold, you can alter your theory to include gravity, extra dimensions, strings, or whatever you want! This is one of the biggest reasons why differential forms are so useful to physicists.