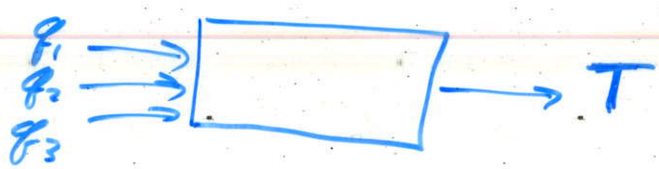
# Displacement Vector $\vec{r}$

Cartesian: $x\hat{e}_x + y\hat{e}_y + z\hat{e}_z = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x, y, z)$

Spherical: $r\hat{e}_r +$

Cylindrical: $s\hat{e}_s + z\hat{e}_z$

# Scalar Fields (Scalar Function of Coordinates)




e.g. Temperature field in the room.

$T(\vec{r}) = T(q_1, q_2, q_3)$ e.g. $T(x, y, z)$ or $T(r, \theta, \varphi)$

↑ Coordinate-free notation

$T(x, y, z) = T(1\text{m east}, 2\text{m north}, 3\text{m up})$

$T(r, \theta, \varphi) = T(4000\text{mi}, 57.3°, 263.2°)$

functions $T(x, y, z) = x^2\sqrt{y} + \cos(z)$

$T(r, \theta, \varphi) = \frac{1}{r}\sec\varphi$

# Vector Fields (Vector Functions of Coordinates)


3 coordinates $\{q_i\}$

3 components $\{V_i\}$

because $\mathbb{R}^3$

$V_\theta(r, \theta, \varphi) = \vec{V} \cdot \hat{e}_\theta$

$$\vec{V}(\vec{r}) = V_1(q_1, q_2, q_3)\,\hat{e}_1(q_1, q_2, q_3) + V_2(q_1, q_2, q_3)\,\hat{e}_2(q_1, q_2, q_3) + V_3(q_1, q_2, q_3)\,\hat{e}_3(q_1, q_2, q_3)$$

Cartesian: $\vec{V}(\vec{r}) = V_x(x,y,z)\,\underbrace{\hat{e}_x}_{\text{constant vector}} + V_y(x,y,z)\,\hat{e}_y + V_z(x,y,z)\,\hat{e}_z$

e.g. $\vec{V}(\vec{r}) = x^3 y^4\,\hat{e}_x + \frac{1}{x}\sin y\,\hat{e}_y + 3\,\hat{e}_z$

---

Spherical: $\vec{V}(\vec{r}) = \underbrace{V_r(r,\theta,\varphi)}_{\text{radial component}}\,\hat{e}_r(r,\theta,\varphi)$

$+ \underbrace{V_\theta(r,\theta,\varphi)}_{\text{polar component}}\,\hat{e}_\theta(r,\theta,\varphi) + \underbrace{V_\varphi(r,\theta,\varphi)}_{\text{azmuthal component}}\,\hat{e}_\varphi(r,\theta,\varphi)$

e.g. $\vec{V}(\vec{r}) = \frac{1}{r}\tan\theta\,\hat{e}_r + \theta^2\,\hat{e}_\theta + r\tan\varphi\cos^2\theta\,\hat{e}_\varphi$

---

3 scalars in a matrix $\neq$ vector

e.g. $\begin{pmatrix} \text{2 apples} \\ \text{1 banana} \\ \text{3 peaches} \end{pmatrix}$

Vectors (in any coordinate system) transform under rotations like $\vec{r}$. vectors: $\vec{v}, \vec{a}, \vec{p}, \vec{L}, \vec{F}, \vec{\tau}$

# Differential Operators

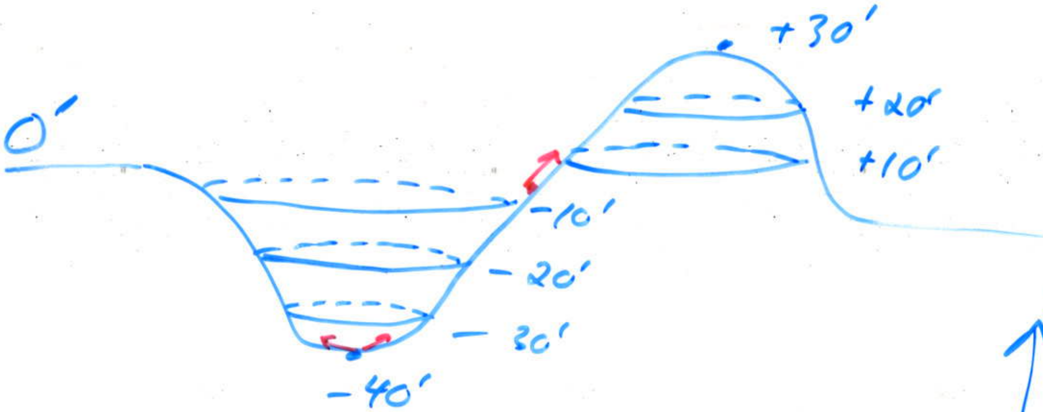
(1) Gradient acts on scalar field and produces a vector field

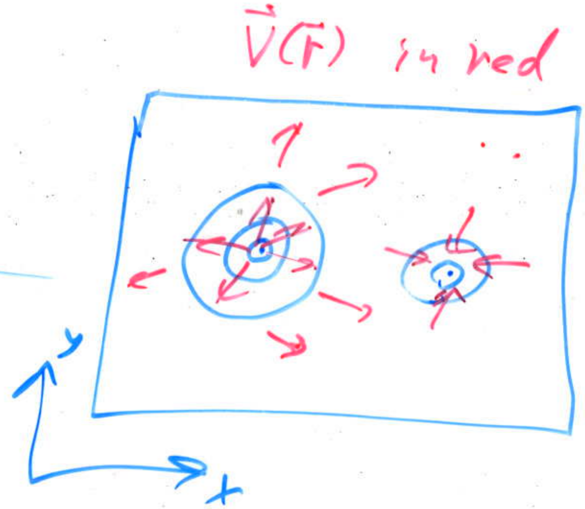
$$T(\vec{r}) \longrightarrow \boxed{\text{Grad}} \rightrightarrows \left.\begin{matrix} V_1(\vec{r}) \\ V_2(\vec{r}) \\ V_3(\vec{r}) \end{matrix}\right\} \vec{V}(\vec{r})$$

$\vec{\nabla}$ nabla, del

$$\vec{V}(\vec{r}) = \vec{\nabla}[T(\vec{r})] = \overrightarrow{\text{grad}}\,[T(\vec{r})]$$

physically $\vec{V}(\vec{r})$ points in the direction of the fastest change in $T(\vec{r})$




Cartesian: $$\vec{\nabla} T_{(\vec{r})} = \hat{e}_x \underbrace{\frac{\partial T(x,y,z)}{\partial x}}_{V_x} + \hat{e}_y \underbrace{\frac{\partial}{\partial y} T(x,y,z)}_{V_y} + \hat{e}_z \underbrace{\frac{\partial}{\partial z} T(x,y,z)}_{V_z}$$

scalar field

$$f(\vec{r}) = x^2 \sin(y) + \cos(z)$$

$$\vec{\nabla} f(\vec{r}) = \hat{e}_x \, 2x \sin(y) + \hat{e}_y \, x^2 \cos(y) + -\hat{e}_z \sin(z)$$

$$= \begin{pmatrix} 2x \sin(y) \\ x^2 \cos(y) \\ -\sin(z) \end{pmatrix}$$

In general $\{q_1, q_2, q_3\}$ $\quad T(\vec{q})$ scalar field

$$\vec{\nabla} T(\vec{q}) = \frac{\hat{e}_1}{h_1} \frac{\partial T}{\partial q_1} + \frac{\hat{e}_2}{h_2} \frac{\partial T}{\partial q_2} + \frac{\hat{e}_3}{h_3} \frac{\partial T}{\partial q_3}$$

spherical polar $\quad \vec{\nabla} T(\vec{r}) = \hat{e}_r \frac{\partial T}{\partial r} + \frac{\hat{e}_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{\hat{e}_\varphi}{r \sin\theta} \frac{\partial T}{\partial \varphi}$

cylindrical polar $\quad \vec{\nabla} T(\vec{r}) = \hat{e}_s \frac{\partial T}{\partial s} + \frac{\hat{e}_\varphi}{s} \frac{\partial T}{\partial \varphi} + \hat{e}_z \frac{\partial T}{\partial z}$

---

e.g. Electrostatic Scalar potential (Voltage)

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2+z^2}}$$

cartesian

point charge $q$ at the origin. in MKS units.

spherical polar $\quad \Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

Next Electrostatic Field $\vec{E}(\vec{r}) \equiv -\vec{\nabla}\Phi(\vec{r})$

Cartesian $\vec{E}(x,y,z) = -\left[\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right] \Phi(x,y,z)$

one piece $\frac{\partial}{\partial x}\Phi(x,y,z) = \frac{\partial}{\partial x} \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2+z^2}}$

$= \frac{q}{4\pi\epsilon_0} \frac{\partial}{\partial x} \left(x^2+y^2+z^2\right)^{-1/2} = \frac{q}{4\pi\epsilon_0} \left(-\frac{1}{2}\right)\left(x^2+y^2+z^2\right)^{-3/2} \cdot 2x$

$= -\frac{q}{4\pi\epsilon_0} \frac{x}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$

---

$$\vec{E}(x,y,z) = \hat{e}_x \frac{q}{4\pi\epsilon_0} \frac{+x}{\left(\sqrt{x^2+y^2+z^2}\right)^2} + \hat{e}_y \frac{q}{4\pi\epsilon_0} \frac{+y}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$+ \hat{e}_z \frac{q}{4\pi\epsilon_0} \frac{+z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}$$

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{\left(\sqrt{x^2+y^2+z^2}\right)^3} = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}}{|\vec{r}|^3} \leftarrow$$

---

Spherical Polar $\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

$$\vec{E}(r,\theta,\varphi) = -\vec{\nabla}\Phi(r,\theta,\varphi) = -\hat{e}_r \frac{\partial \Phi}{\partial r} = -\hat{e}_r \frac{d}{dr}\left[\frac{q}{4\pi\epsilon_0} \frac{1}{r}\right]$$

$$= +\hat{e}_r \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \leftarrow \qquad \hat{e}_r = \frac{\vec{r}}{|\vec{r}|}$$