Transformation of tensors

rank-0 tensor

scalar: T (temperature), mass m

rank-1 tensor

vector: $\vec{V}$ ← coordinate free

$\{V_i\} = (V_1, V_2, V_3)$ component form

"tensors": $\theta_{\mu\nu}$, ϵ_{ijk} Levi-Civita tensor density fully antisymmetric

$$\epsilon_{123} \equiv +1 = \epsilon_{231} = \epsilon_{312}$$

$$\epsilon_{321} \equiv -1 = \epsilon_{213} = \epsilon_{132}$$

$$\epsilon_{112} = 0 \qquad \epsilon_{222} = 0 \qquad \ldots$$

I_{ij} inertia tensor

<u>Not</u> a vector

$$\begin{pmatrix} \text{1 banana} \\ \text{2 apples} \\ \text{5 pears} \end{pmatrix}$$

Scalar T one coordinate system

T' transformed coor's sys.

$$T = T'$$

$\vec{V}$ original coord sys

$\vec{V}'$ transformed

$$\vec{V}' = \underline{\underline{R}} \cdot \vec{V} \quad \text{- coordinate free}$$

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

e.g. $v_1' = \underbrace{r_{11} v_1}_{j=1} + \underbrace{r_{12} v_2}_{j=2} + \underbrace{r_{13} v_3}_{j=3}$

index notation

$$v_i' = \sum_{j=1}^{3} r_{ij} v_j$$

i - free index

j - dummy index

$i = 1$ $\sum_{k=1}^{3} r_{ik} v_k$

recover orange eq.

rank-2 and higher order tensors

$$I'_{ij} = \sum_{k=1}^{3} \sum_{l=1}^{3} r_{ik}\, r_{jl}\, I_{kl}$$

i - free
j - free

Think about tensors like this

$$I'_{ij} \simeq v'_i\, v'_j$$

$$v'_i = \sum_{k=1}^{3} r_{ik}\, v_k \quad \Big/ \quad v'_j = \sum_{l=1}^{3} r_{jl}\, v_l$$

$$I'_{ij} = v'_i v'_j = \sum_{k=1}^{3} \sum_{l=1}^{3} r_{ik}\, r_{jl}\, \overset{I_{kl}}{\boxed{v_k\, v_l}}$$

Fields any quantity that changes with position in space.

scalar field $T(\vec{r})$, $T(x, y, z)$

$T(r, \theta, \varphi)$

vector field $\vec{V}(\vec{r}) = \begin{pmatrix} V_x(x, y, z) \\ V_y(x, y, z) \\ V_z(x, y, z) \end{pmatrix}$

e.g.

$$\vec{V}(\vec{r}) = \begin{pmatrix} x^2 + 2yz \\ 0 \\ xyz + \sqrt{y} \end{pmatrix}$$

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tensor field $F_{ij}(x, y, z)$ 9 components

$F_{11}(x, y, z)$

$F_{12}(x, y, z)$

$\vdots$

$F_{33}(x, y, z)$

eg $F_{11} = 2 + 7x$

$F_{12} = x^2 + yz$

$F_{33} = \sin(y)$

Differential Operator

del : $\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$ in Cartesian only.

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$= \hat{e}_i \frac{\partial}{\partial r_i}$$

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gradient acts on a scalar function and returns a vector function

$$\vec{\nabla} T(x,y,z) = \vec{V}(x,y,z) \begin{cases} V_x(x,y,z)\,\hat{\imath} \\ V_y(x,y,z)\,\hat{\jmath} \\ V_z(x,y,z)\,\hat{k} \end{cases}$$

grad $T(x,y,z)$

$$\vec{\nabla} T(\vec{r}) = \hat{\imath}\frac{\partial T}{\partial x} + \hat{\jmath}\frac{\partial T}{\partial y} + \hat{k}\frac{\partial T}{\partial z}$$

$$= \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right)$$