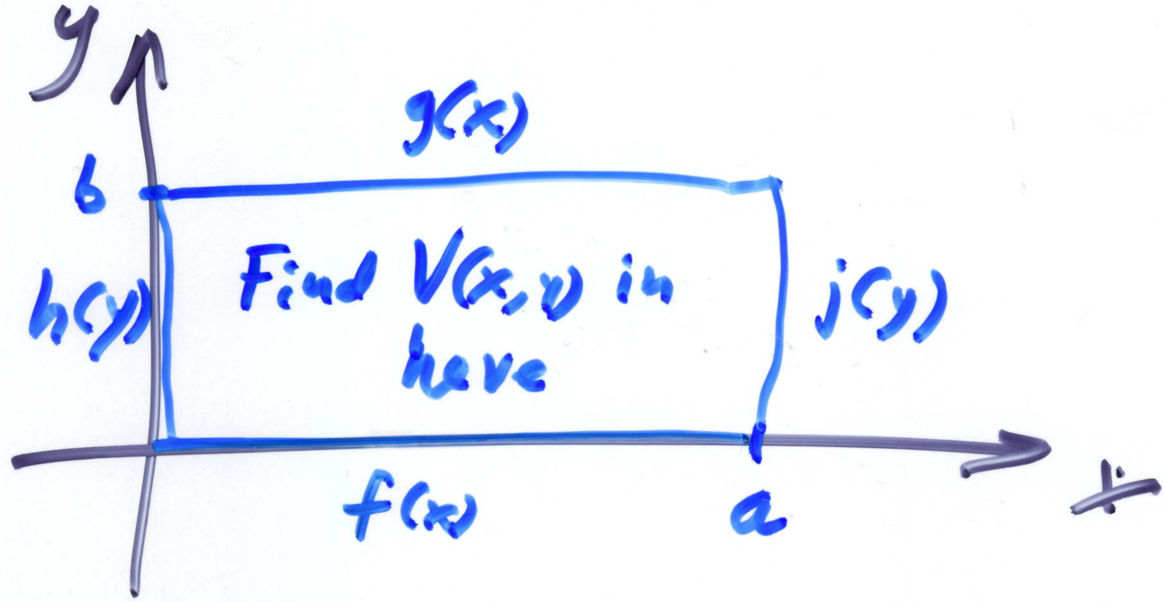
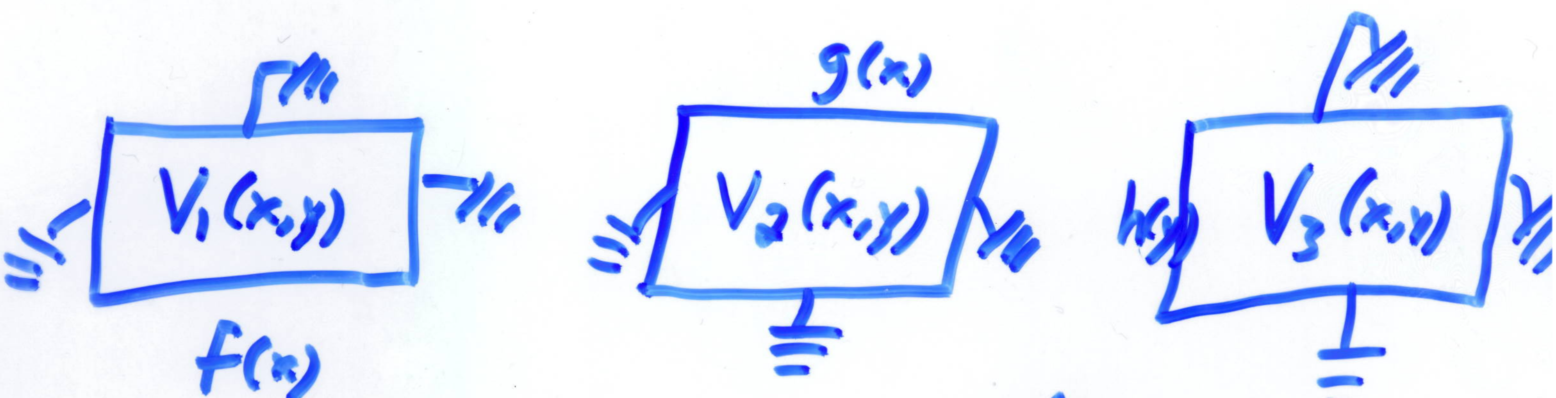
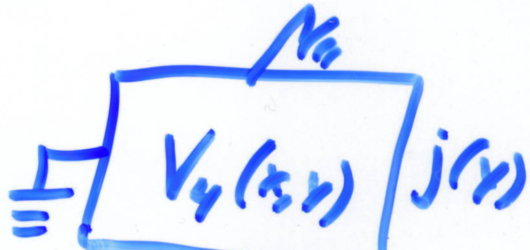
Most general problem



Use superposition:





$\nabla^2 V_1(x,y) = 0$

$V_1(0,y) = 0$ left

$V_1(a,y) = 0$ right

$V_1(x,b) = 0$ top

$V_1(x,0) = f(x)$ bottom

$$V(x,y) = V_1 + V_2 + V_3 + V_4$$

$$V_1(x,y) = \left[K_1 \sin(\alpha x) + \cancel{K_2}^{\,0} \cos(\alpha x)\right] \cdot$$

$$\cdot \left[K_3 \sinh(\alpha y) + K_4 \cosh(\alpha y)\right]$$

Satisfies $\nabla^2 V_1(x,y) = 0$ by construction. Now choose $k_1\, k_2\, k_3\, k_4$ to satisfy the boundary conditions (B.C.)

B.C. at $x = 0 \Rightarrow V_1(0,y) = 0$

$$V_1(0,y) = k_2 \left[k_3 \sinh(\alpha y) + k_4 \cosh(\alpha y)\right] = 0$$

$$\Rightarrow k_2 = 0$$

B.C. at $x = a \Rightarrow V_1(a,y) = 0$

$$V_1(a,y) = k_1 \underline{\sin(\alpha a)} \left[k_3 \sinh \alpha y + k_4 \cosh \alpha y\right] = 0$$

$$\Rightarrow \alpha a = n\pi \quad , \quad n = 1, 2, 3, \ldots$$

$$\alpha = \frac{n\pi}{a}$$

Solution so far

$$V_1(x,y) = k_1 \sin\left(\frac{n\pi x}{a}\right) \cdot$$

$$\cdot \left[k_3 \sinh\left(\frac{n\pi y}{a}\right) + k_4 \cosh\left(\frac{n\pi y}{a}\right)\right]$$

B.C. for $y=b$ (top): $V_1(x,b) = 0$

$$V_1(x,b) = k_1 \sin\left(\frac{n\pi x}{a}\right) \cdot$$

$$\cdot \left[k_3 \sinh\left(\frac{n\pi b}{a}\right) + k_4 \cosh\left(\frac{n\pi b}{a}\right)\right] = 0$$

$$\Rightarrow k_4 = -k_3 \frac{\sinh\left(\frac{n\pi b}{a}\right)}{\cosh\left(\frac{n\pi b}{a}\right)} = -k_3 \tanh\left(\frac{n\pi b}{a}\right)$$

Solution so far:

$$V_1(x,y) = k_1 \sin\left(\frac{n\pi x}{a}\right) \cdot k_3 \cdot$$

$$\cdot \left[\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right) \cosh\left(\frac{n\pi y}{a}\right)\right]$$

$$k_1 \times k_3 \equiv A_n$$

$$V_1(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\left[\sinh\left(\frac{n\pi y}{a}\right) - \tanh\left(\frac{n\pi b}{a}\right)\cdot\cosh\left(\frac{n\pi y}{a}\right)\right]$$

Laplace: $\nabla^2 V_1(x,y) = 0$

B.C. on left, right, top.

Use A_n to satisfy the B.C. on bottom.

$$V_1(x,0) = f(x)$$

$\sinh(0) = 0$
$\cosh(0) = 1$

$$= \sum_{n=1}^{\infty} -A_n \sin\left(\frac{n\pi x}{a}\right)\tanh\left(\frac{n\pi b}{a}\right)$$

$$\int_0^a \frac{2}{a}\sin\left(\frac{p\pi x}{a}\right) f(x)\,dx = \int_0^a \frac{2}{a}\sin\left(\frac{p\pi x}{a}\right)\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\tanh\left(\frac{n\pi b}{a}\right)dx$$

$$-\frac{2}{a}\int_{x=0}^{a} \sin\left(\frac{p\pi x}{a}\right) f(x)\,dx = \sum_{n=1}^{\infty} A_n \delta_{np} \tanh\left(\frac{n\pi b}{a}\right)$$

$$= A_p \tanh\left(\frac{p\pi b}{a}\right)$$

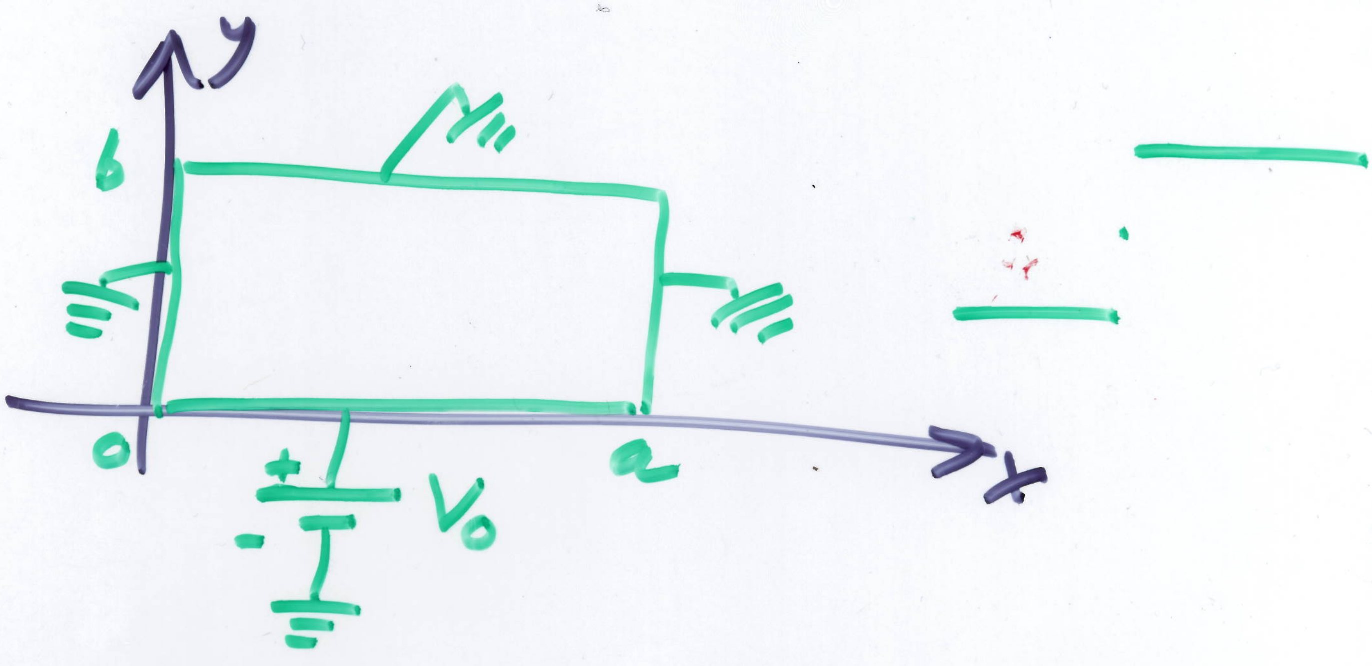
$$\frac{2}{a}\int_0^a \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)dx = \begin{cases} 0 & \text{if } n\neq p \\ 1 & \text{if } n=p \end{cases}$$

Fourier's trick

$$= \delta_{np}$$

$$A_p = \frac{\frac{2}{a}\int_{x=0}^{a} \sin\left(\frac{p\pi x}{a}\right) f(x)\, dx}{\tanh\left(\frac{p\pi b}{a}\right)}$$

Special case: $f(x) = \text{constant} = V_0$



$$A_p = \frac{\frac{2}{a}\int_{x=0}^{a} \sin\left(\frac{p\pi x}{a}\right) V_0 \, dx}{\tanh\left(\frac{p\pi b}{a}\right)}$$

$$= \frac{-2}{\cancel{a}} V_0 \cos\left(\frac{p\pi x}{a}\right)\Bigg|_{x=0}^{a} \frac{\cancel{a}}{p\pi} \frac{1}{\tanh\left(\frac{p\pi b}{a}\right)}$$

$$= \frac{-2V_0}{p\pi \tanh\left(\frac{p\pi b}{a}\right)} \left[\underbrace{\cos(p\pi)}_{\substack{+1,\ p \text{ even} \\ -1,\ p \text{ odd}}} - \cos(0)\right]$$

$$= \frac{-2V_0}{p\pi \tanh\left(\frac{p\pi b}{a}\right)} \left[(-1)^p - 1\right]$$

$$A_p = \begin{cases} \dfrac{+4V_0}{p\pi \tanh\left(\frac{p\pi b}{a}\right)}, & p \text{ odd} \\ 0, & p \text{ even} \end{cases}$$

We had in 2 dimensions:

$$\left.\begin{aligned} \frac{C''(x)}{C(x)} &= -\alpha^2 \\ \frac{D''(y)}{D(y)} &= +\alpha^2 \end{aligned}\right\} \text{Why these signs?}$$

The solution $V(x,y)$ must equal zero at $x=0$ <u>and</u> at $x=a$

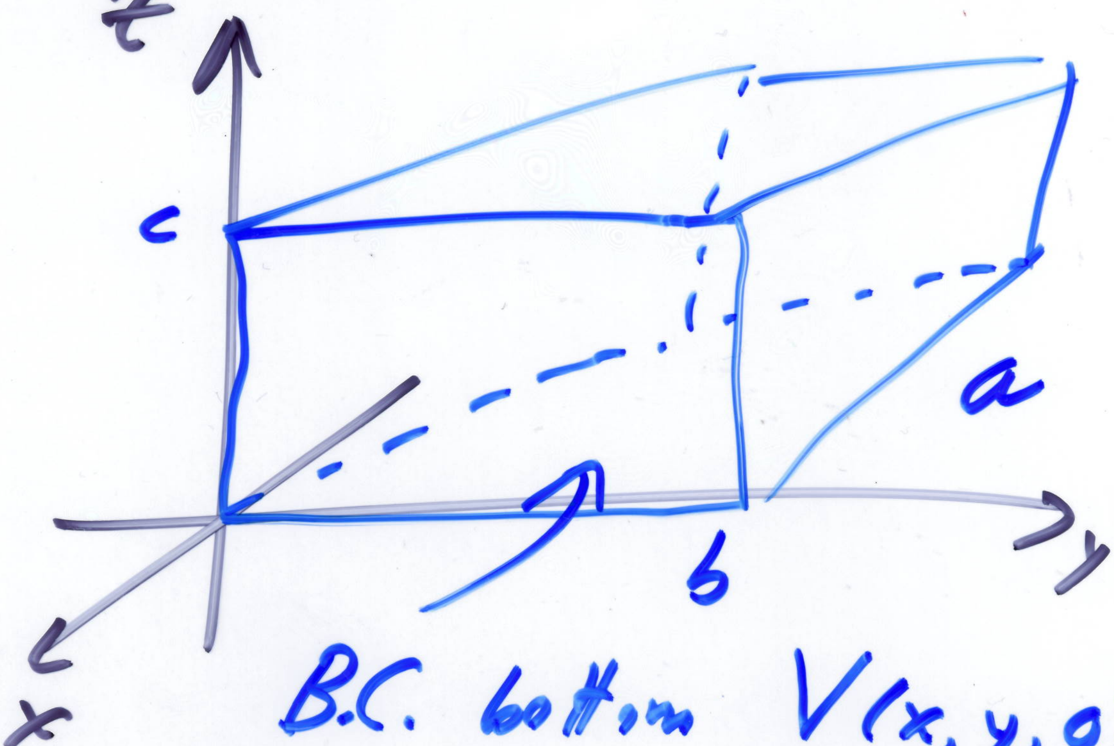
Sines + cosines can vanish in ∞'ly many places while

sinh + cosh can vanish in one or zero places.

Also, we want to build the function $f(x)$ (B.C. on bottom) out of our solutions.

sines + cosines are complete.

sinh + cosh are not complete.



B.C. bottom $V(x,y,0) = f(x,y)$
other faces grounded

$$\nabla^2 V(x,y,z) = 0$$

$$V(x,y,z) = C(x) \cdot D(y) \cdot F(z)$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) C(x)\, D(y)\, F(z) = 0$$

$$0 = C''(x)\, D(y) F(z) + C(x)\, D''(y) F(z) + C(x) D(y) F''(z)$$

$$0 = \frac{C''(x)}{C(x)} + \frac{D''(y)}{D(y)} + \frac{F''(z)}{F(z)}$$

$$\left.\begin{aligned} \frac{C''(x)}{C(x)} &= -\alpha^2 \\[2em] \frac{D''(y)}{D(y)} &= -\beta^2 \end{aligned}\right\} \text{two separation constants}$$

$$\frac{F''(z)}{F(z)} = +\alpha^2 + \beta^2$$

$C''(x) + \alpha^2 C(x) = 0$ $\quad\quad$ $\sin, \cos(\alpha x)$

$D''(y) + \beta^2 D(y) = 0$ $\quad\quad$ $\sin, \cos(\beta y)$

$\underline{F''(z)} - (\alpha^2 + \beta^2) F(z) = 0$ $\quad\quad$ $\sinh, \cosh($

$$\sinh\left(\sqrt{\alpha^2+\beta^2}\, z\right)$$

$$\cosh(\quad\quad\quad\quad)$$