For problems with azimuthal symmetry (no ϕ dependence) <u>and</u> with the North + South Poles included ($\theta = 0$, $\theta = \pi$).

$$V(r,\theta) = R(r) \cdot T(\theta)$$

$$= \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)$$

Laurent series:

$$B_1 r^{-2} \quad + B_0 r^{-1} \quad + A_0 r^0 \quad + A_1 r^1$$

$$C_{-2} r^{-2} \quad + C_{-1} r^{-1} \quad + C_0 r^0 \quad + C_1 r^1$$

If you are solving the interior of a sphere and the origin $r=0$ is included:

$$B_0 = 0 \quad , \quad B_1 = 0 \quad \ldots$$

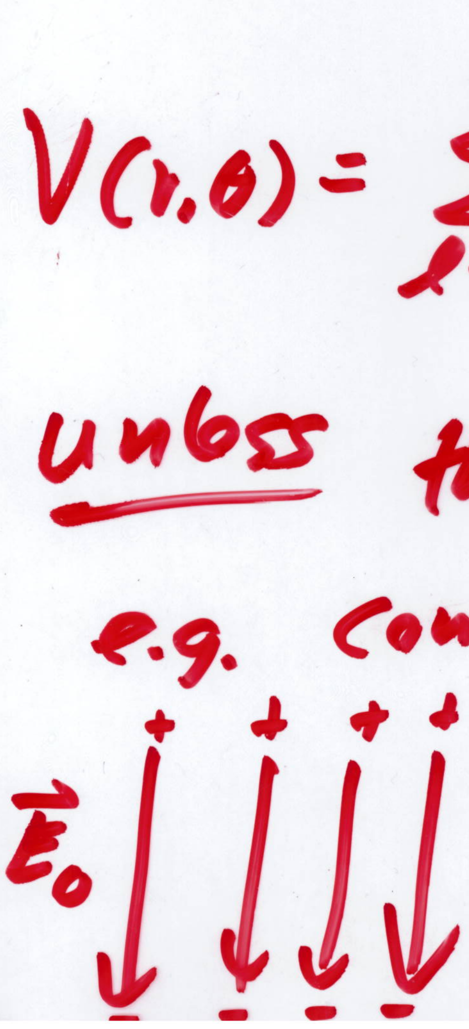
$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

If you are solve the exterior problem $r=\infty$ included

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l \cos\theta$$

<u>unless</u> there is charge at ∞.

e.g. constant Electric field



$$V = E_0 z = E_0 r\cos\theta$$
$$= A_1 r^1 P_1(\cos\theta)$$

A <u>remarkable</u> result:

If you know the potential on the positive z-axis:

$$V(z) \quad \text{for} \quad 0 \le z \le \infty$$

then you can figure out the potential everywhere $V(\vec{r})$

$$V(r, \theta=0) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right] \cdot 1$$

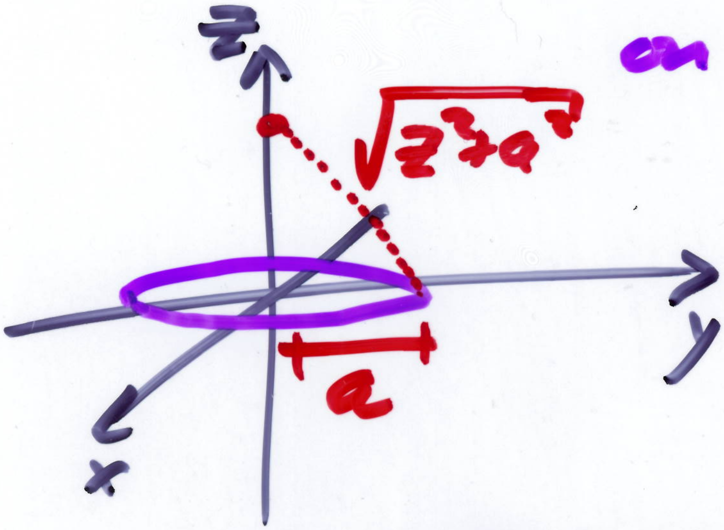
that is a Laurent expansion.

$$\cdots \frac{B_1}{z^2} + \frac{B_0}{z} + A_0 + A_1 z^1 + A_2 z^2 \cdots$$

Remember $P_\ell(1) = 1$

$\theta = 0$
$\cos\theta = 1$
North Pole

E.g. Ring of charge $\underline{q}$, radius $\underline{a}$. Find the voltage on the z-axis.



$$V(z) = \frac{kq}{\sqrt{z^2+a^2}}$$

For $z > a$

$$V(z) = \frac{kq}{z}\frac{1}{\sqrt{1+\frac{a^2}{z^2}}} = \frac{kq}{z}\left[1+\frac{a^2}{z^2}\right]^{-\frac{1}{2}}$$

$$= \frac{kq}{z}\sum_{\ell=0}^{\infty}(-1)^{\ell}\left[\frac{(2\ell-1)!!}{2^{\ell}\,\ell!}\right]\left(\frac{a^2}{z^2}\right)^{\ell}$$

$$x!! = x(x-2)(x-4)\cdots$$

$$x! = x(x-1)(x-2)\cdots 1$$

$V(r,\theta)$ = Take $V(z)$ replace $z \to r$, multiply by $P_\ell(\cos\theta)$

"Going off axis!"

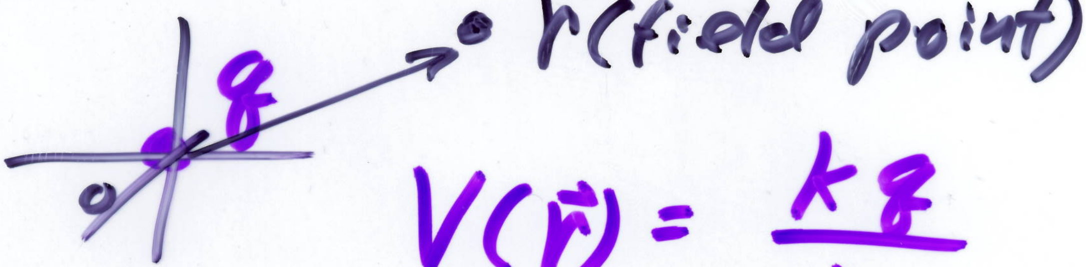
$$V(r,\theta) = \frac{kq}{r}\sum_{\ell=0}^{\infty}(-1)^\ell\left[\frac{(2\ell-1)!!}{2^\ell\,\ell!}\right]\left(\frac{a^2}{r^2}\right)^\ell P_{2\ell}(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty}\underbrace{\left[kq(-1)^\ell\left[\frac{(2\ell-1)!!}{2^\ell\,\ell!}\right]a^{2\ell}\right]}_{B_{2\ell}}\frac{1}{r^{2\ell+1}}P_{2\ell}(\cos\theta)$$

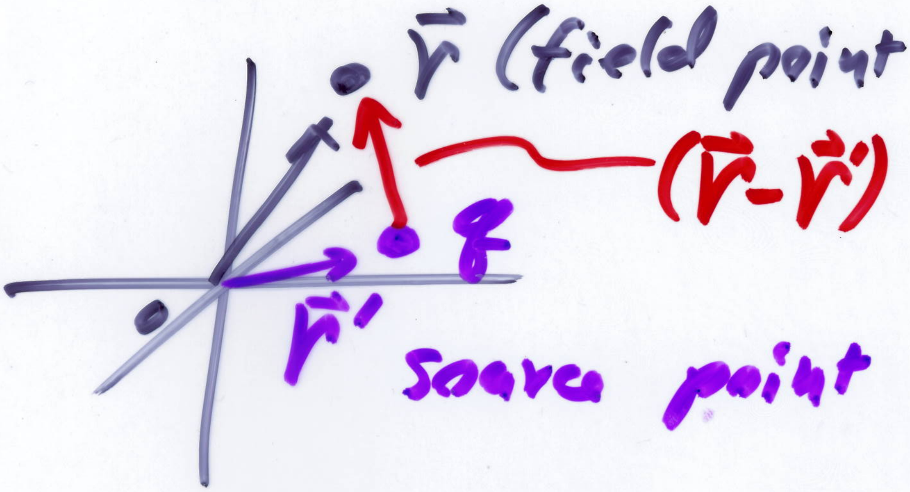
Multipole expansion:

pole = monopole = point charge.

monopole potential

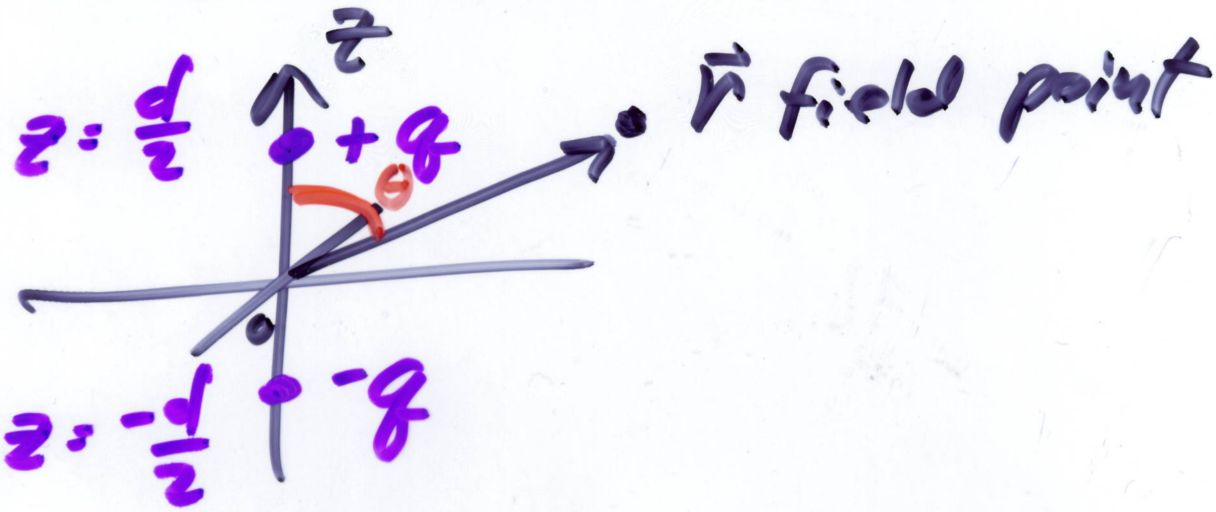


$$V(\vec{r}) = \frac{kq}{r}$$



$$V(\vec{r}) = \frac{kq}{|\vec{r}-\vec{r}'|}$$

Dipole potential



charge density

$$\rho(\vec{r}') = q\,\delta(x')\delta(y')\delta(z'-\tfrac{d}{2}) - q\,\delta(x')\delta(y')\delta(z'+\tfrac{d}{2})$$

$$V(\vec{r}) = \iiint_{\substack{\text{All}\\ \text{space}}} \frac{k\,\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

$$= \frac{kq}{|\vec{r}-\frac{d}{2}\hat{e}_3|} - \frac{kq}{|\vec{r}+\frac{d}{2}\hat{e}_3|}$$

$$\frac{1}{\left|\vec{r} \mp \frac{d}{2}\hat{e}_3\right|} = \frac{1}{\sqrt{\left(\vec{r} \mp \frac{d}{2}\hat{e}_3\right)\cdot\left(\vec{r} \mp \frac{d}{2}\hat{e}_3\right)}}$$

$$= \frac{1}{\sqrt{r^2 \mp rd\cos\theta + \frac{d^2}{4}}}$$

$$= \left(r^2 \mp rd\cos\theta + \frac{d^2}{4}\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta + \text{higher order}\right)$$

$$V(r) = \frac{kq}{r}\left[\frac{d}{r}\cos\theta + \text{higher}\right]$$