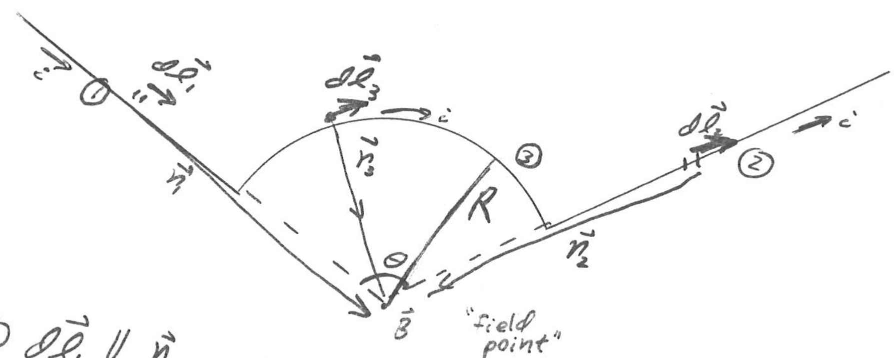
Biot-Savart Examples

Ex Circular Arc



① $d\vec{l}_1 \parallel \vec{r}_1 \Rightarrow d\vec{l}_1 \times \vec{r}_1 = 0$

② $d\vec{l}_2 \parallel \vec{r}_2 \Rightarrow d\vec{l}_2 \times \vec{r}_2 = 0$

Biot-Savart Law: $d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{l} \times \vec{r}}{r^3}$

③ $d\vec{l}_3 \perp \vec{r}_3 \Rightarrow |d\vec{l}_3 \times \vec{r}_3| = |dl|\,|r| \sin\frac{\pi}{2}$

Direction of $\vec{B}$ is ⊗ into page

Magnitude

$dB = \frac{\mu_0}{4\pi} \frac{i\, dl\, r}{r^3}$ $\qquad r = R$, $dl = R\,d\theta$

$dB = \frac{\mu_0}{4\pi} \frac{i\, d\theta}{R}$

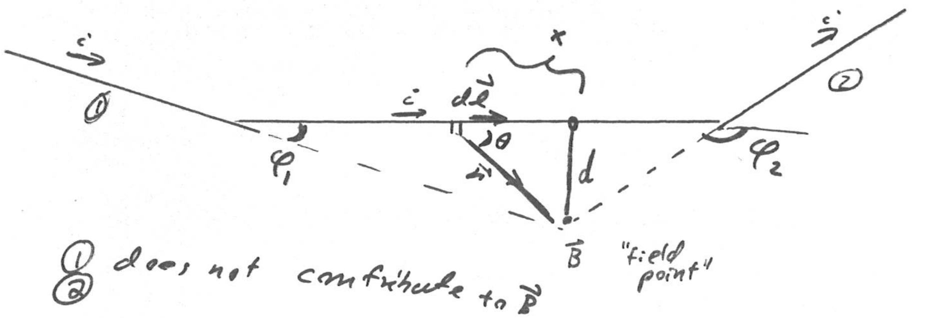
$$B = \int_0^\theta dB = \int_0^\theta \frac{\mu_0}{4\pi} \frac{i\, d\theta}{R}$$

$$\frac{\mu_0 i}{4\pi R} \int_0^\theta d\theta = \boxed{\frac{\mu_0}{4\pi} \frac{i\theta}{R}}$$

$\theta \to 2\pi$ full circle

$$B = \frac{\mu_0}{4\pi} \frac{i\, 2\pi}{R}$$

Ex: Magnetic field due to a <u>finite</u> length of current-carrying wire.



① ② does not contribute to $\vec{B}$

③ directions ⊗ into page

<u>magnitude</u>

Biot-Savart $\quad d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \vec{r}}{r^3}$

$|d\vec{\ell}| = dx$

$|d\vec{\ell} \times \vec{r}| = dx\, \underline{r} \sin\theta$

$dB = \frac{\mu_0}{4\pi} \frac{i\, dx \sin\theta}{r^2}$

$d = r\sin\theta \Rightarrow r = \frac{d}{\sin\theta}$

$dB = \frac{\mu_0}{4\pi} \frac{i\, dx \sin^3\theta}{d^2}$

$X = \frac{-d}{\tan\theta}$ $\qquad \tan\theta = \frac{opp}{adj} = \frac{d}{X}$

$dx = \frac{+d}{\sin^2\theta}\, d\theta$ $\qquad \frac{d}{d\theta}\cot\theta = -\csc^2\theta$

$\frac{d}{d\theta}\left(\frac{1}{\tan\theta}\right) = \frac{-1}{\sin^2\theta}$

$dB = \frac{\mu_0}{4\pi}\, i \left(\frac{d}{\sin^2\theta}\right) \frac{\sin^3\theta}{d^2} d\theta = \frac{\mu_0}{4\pi} \frac{i \sin\theta}{d}\, d\theta$

$B = \int dB = \int_{\theta=\varphi_1}^{\varphi_2} \frac{\mu_0}{4\pi} \frac{i \sin\theta\, d\theta}{d}$

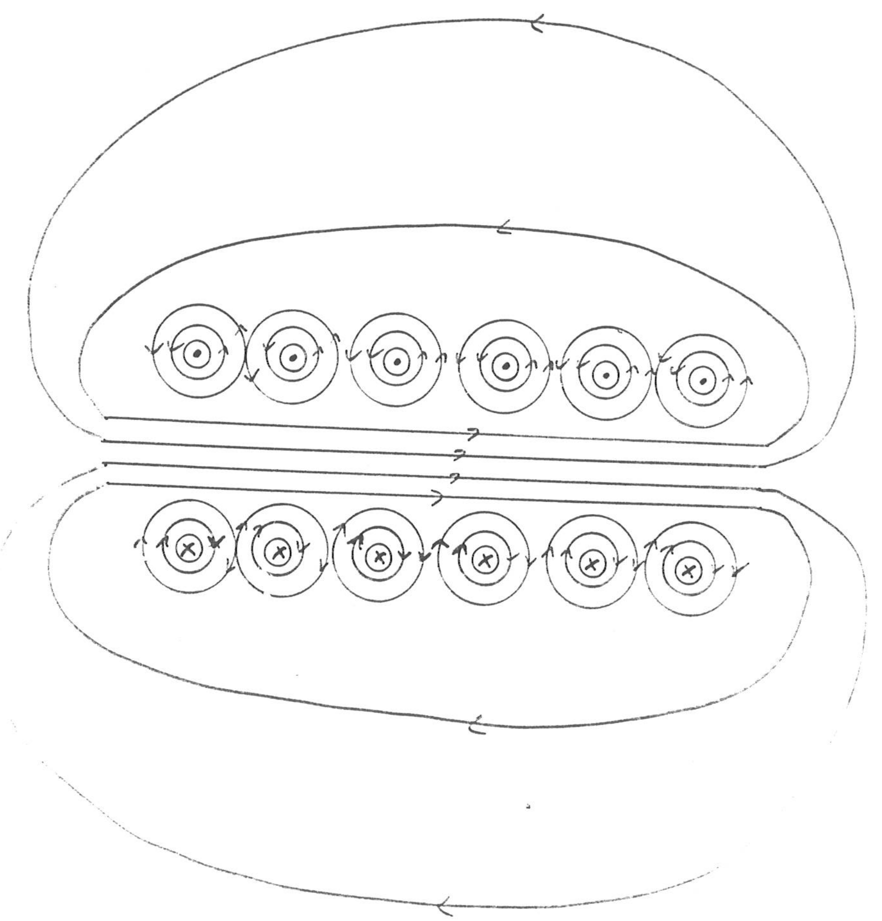
$B = \frac{\mu_0}{4\pi} \frac{i}{d} \int_{\varphi_1}^{\varphi_2} \sin\theta\, d\theta$

$B = \frac{\mu_0}{4\pi} \frac{i}{d} \left[\cos\varphi_1 - \cos\varphi_2\right]$

Check: Infinite wire $\varphi_1 = 0 \quad \varphi = 180$

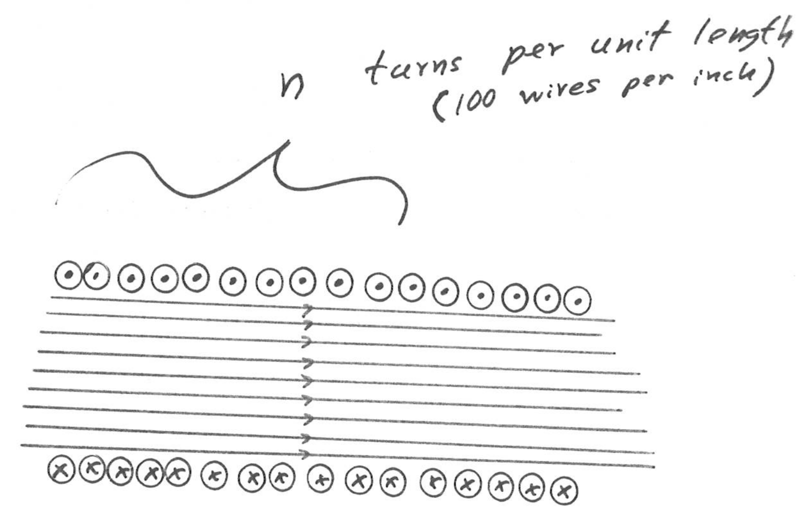
$B = \frac{\mu_0}{4\pi} \frac{i}{d} [2]$ ✓

The Solenoid



$\vec{B}$ field lines are <u>closed</u> loops.

The Solenoid



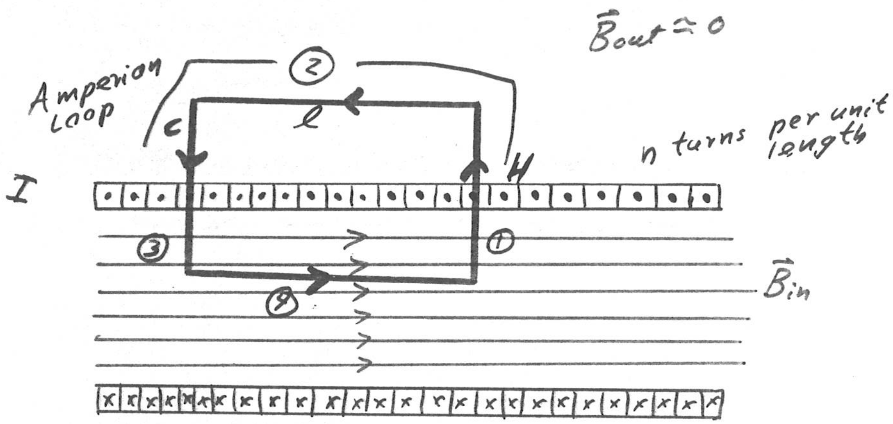
If the solenoid is very long compared to its radius and if the coils are closely spaced then:

$\vec{B}_{inside} \approx$ constant

$\vec{B}_{outside} \approx 0$

Well, not really, but the $\vec{B}$ field is <u>much</u> less dense outside.

Magnetic field inside a solenoid by Ampere's Law:



Ampere's Law: $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 \, i_{enc}$

$$= \underbrace{\int_{①} \vec{B}\cdot d\vec{\ell}}_{\vec{B}\perp d\vec{\ell}} + \underbrace{\int_{②} \vec{B}\cdot d\vec{\ell}}_{B=0} + \underbrace{\int_{③} \vec{B}\cdot d\vec{\ell}}_{\vec{B}\perp d\vec{\ell}} + \int_{④} \vec{B}\cdot d\vec{\ell}$$

$$= \int_{④} \vec{B}\cdot d\vec{\ell} = |B| \int d\ell = B\ell$$

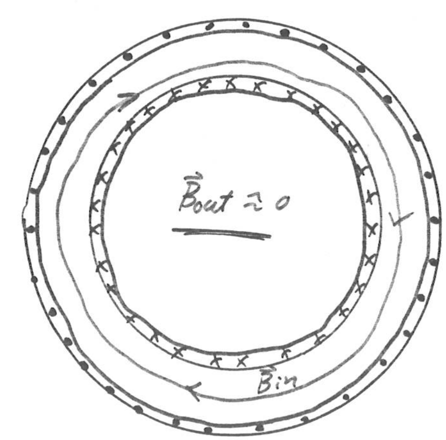
$$= \mu_0 \, i_{enc} = I\mu_0 \boxed{n\ell} \leftarrow \text{\# wires carrying } I$$

$$B\cancel{\ell} = \mu_0 I n \cancel{\ell} \qquad \boxed{B_{in} = \mu_0 I n}$$

The Toroid

Instead of making the solenoid infinitely long to get a very small $\vec{B}$ field outside, one can attach the open ends to each other to make a doughnut shape.

Total of N turns

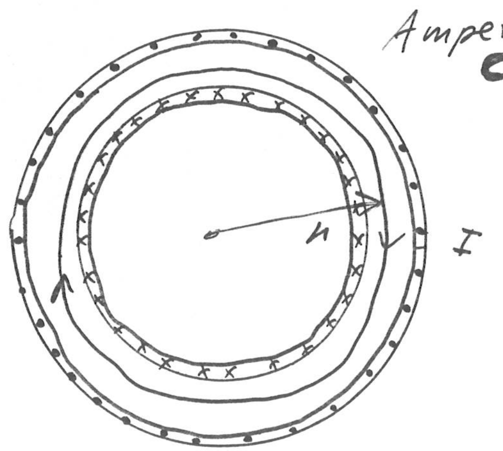


$\vec{B}_{out} \approx 0$

This time, $\vec{B}_{in} \neq$ constant.

The Toroid

Instead of making the solenoid infinitely long to get a very small $\vec{B}$ field outside, one can attach the open ends to each other to make a doughnut shape.



Ampere's law: $\oint_c \vec{B} \cdot d\vec{l} = \mu_0 i_{enc}$

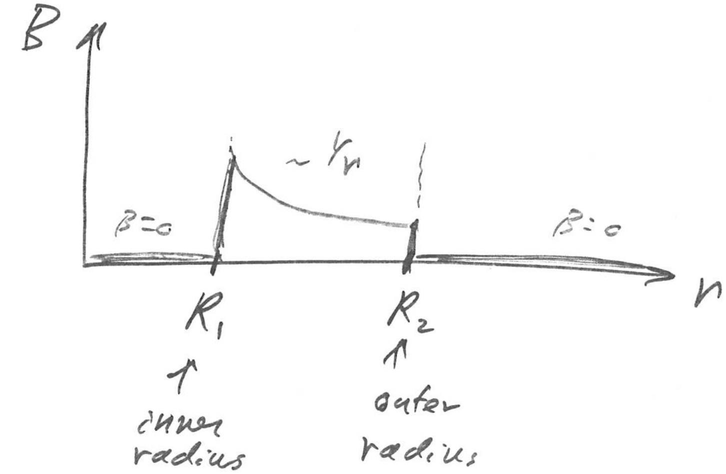
$\vec{B} \parallel d\vec{l}$ on Amperian Loop C

$\oint_c |B|\,|dl| = B \oint_c dl = B(2\pi r)$

$= \mu_0 i_{enc} = \mu_0 (NI)$

$B_{inside} = \dfrac{\mu_0 NI}{2\pi r} = \boxed{\dfrac{\mu_0}{4\pi} \dfrac{2NI}{r}}$

$B_{outside} = 0$



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n = wires per unit lengt

$= \dfrac{N}{2\pi r}$

$B_{toroid} = \dfrac{\mu_0}{4\pi} \dfrac{2I}{r} (2\pi r n) = \mu_0 I n$

$= B_{solenoid}$