Gauss' Law

intinitesimal vector that points out

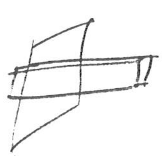
$$\oiint_S \vec{E} \cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0}$$

evaluated on the closed surface S

closed surface

spherical Gaussian surface

cylinder



pillbox

Ampere's Law

intinitesimal line element points along the curve C

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 \, i_{enc}$$

evaluated on closed curve C

current within the closed curve C

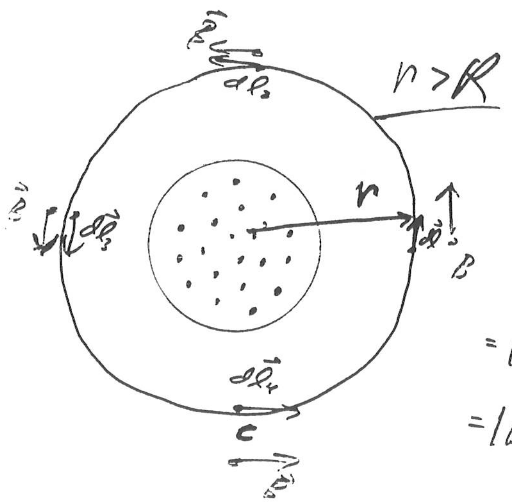
closed curve

"Amperean Loops"

Useful in cases of high symmetry

Ex.

A straight wire of radius R carries a current I distributed uniformly over its cross-sectional area. Find the magnetic field everywhere.



$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i_{enc} = \mu_0 I$$

$$= \oint_C |B|\,|d\ell|$$

$$= |B| \oint_C d\ell$$

$$= |B|\, 2\pi r = \mu_0 I$$

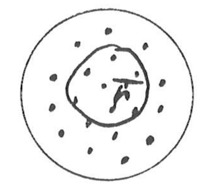
$$\therefore\ |B| = \frac{\mu_0 I}{2\pi r}$$

$$= \left(\frac{\mu_0}{4\pi}\right) \frac{2I}{r}$$

Ex.

A straight wire of radius R carries a current I distributed uniformly over its cross-sectional area. Find the magnetic field everywhere.

$r < R$

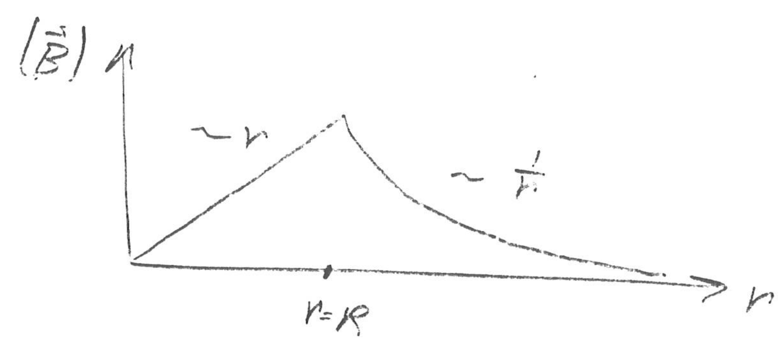


$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i_{enc}$$

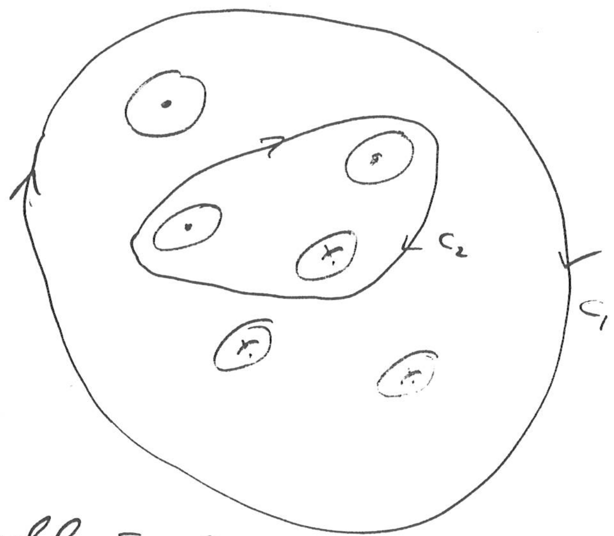
$$|B| \oint_C d\ell = \mu_0 \left(\frac{I}{R^2} r^2\right)$$

$$|B|\, 2\pi r = \frac{\mu_0 I r^2}{R^2}$$

$$\therefore\ |B| = \left(\frac{\mu_0}{4\pi}\right) \frac{2Ir}{R^2}$$



Ex



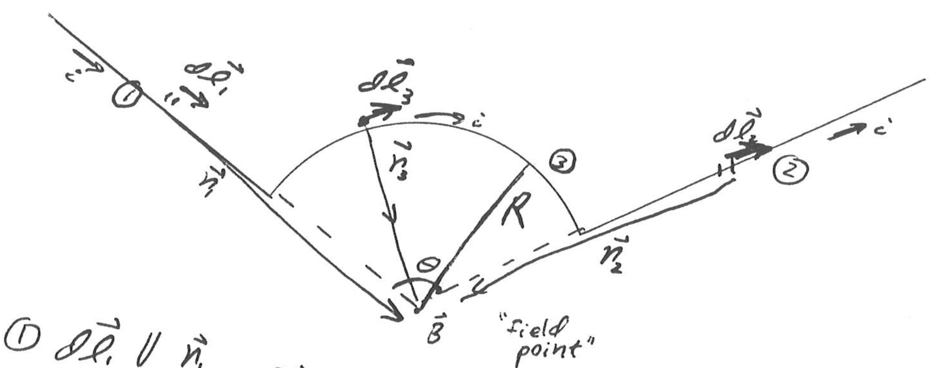
$$\oint_{C_1} \vec{B}\cdot d\ell = 0$$

$$\oint_{C_2} \vec{B}\cdot d\vec{\ell} = \mu_0 I = \mu_0 (I + I - I)$$

If the direction of C_2 is reversed then $i_{enc} = -I = (-I - I + I)$

Biot-Savart Examples

Ex Circular Arc



① $d\vec{\ell}_1 \parallel \vec{r}_1 \Rightarrow d\vec{\ell}_1 \times \vec{r}_1 = 0$

② $d\vec{\ell}_2 \parallel \vec{r}_2 \Rightarrow d\vec{\ell}_2 \times \vec{r}_2 = 0$

Biot-Savart Law: $d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \vec{r}}{r^3}$

③ $d\vec{\ell}_3 \perp \vec{r}_3 \Rightarrow |d\vec{\ell}_3 \times \vec{r}_3| = |d\ell|\,|r|\,\cancel{\sin\theta}\,1$

Direction of $\vec{B}$ is $\otimes$ into page

Magnitude

$$dB = \frac{\mu_0}{4\pi} \frac{i\, d\ell\, r}{r^3} \qquad r = R$$

$$dB = \frac{\mu_0}{4\pi} \frac{i\, d\theta}{R} \qquad d\ell = R\, d\theta$$

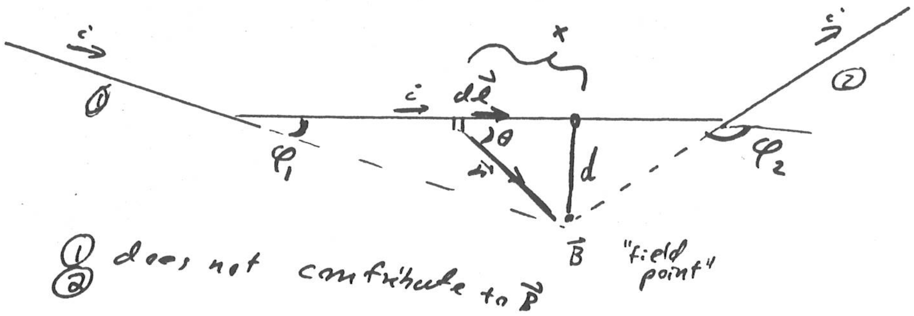
$$B = \int_0^\theta dB = \int_0^\theta \frac{\mu_0}{4\pi} \frac{i\, d\theta}{R}$$

$$\frac{\mu_0 i}{4\pi R} \int_0^\theta d\theta = \boxed{\frac{\mu_0}{4\pi} \frac{i\theta}{R}}$$

$\theta \rightarrow 2\pi$ full circle

$$B = \frac{\mu_0}{4\pi} \frac{i\, 2\pi}{R}$$

Ex: Magnetic field due to a finite length of current-carrying wire.



① ② does not contribute to $\vec{B}$

③ directions ⊗ into page

magnitude

Biot-Savart $\quad d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{l} \times \vec{r}}{r^3}$

$|d\vec{l}| = dx$

$|d\vec{l} \times \vec{r}| = dx\, r \sin\theta$

$dB = \frac{\mu_0}{4\pi} \frac{i\, dx \sin\theta}{r^2}$

$d = r \sin\theta \Rightarrow r = \frac{d}{\sin\theta}$

$dB = \frac{\mu_0}{4\pi} \frac{i\, dx \sin^3\theta}{d^2}$

$X = \frac{-d}{\tan\theta} \qquad \tan\theta = \frac{opp}{adj} = \frac{d}{x}$

$dx = \frac{+d}{\sin^2\theta}\, d\theta \qquad \frac{d}{d\theta} \cot\theta = -\csc^2\theta$

$\frac{d}{d\theta}\left(\frac{1}{\tan\theta}\right) = \frac{-1}{\sin^2\theta}$

$dB = \frac{\mu_0}{4\pi}\, i \left(\frac{d}{\sin^2\theta}\right) \frac{\sin^3\theta}{d^2} d\theta = \frac{\mu_0}{4\pi} \frac{i \sin\theta}{d}\, d\theta$

$B = \int dB = \int_{\theta=\varphi_1}^{\varphi_2} \frac{\mu_0}{4\pi} \frac{i \sin\theta\, d\theta}{d}$

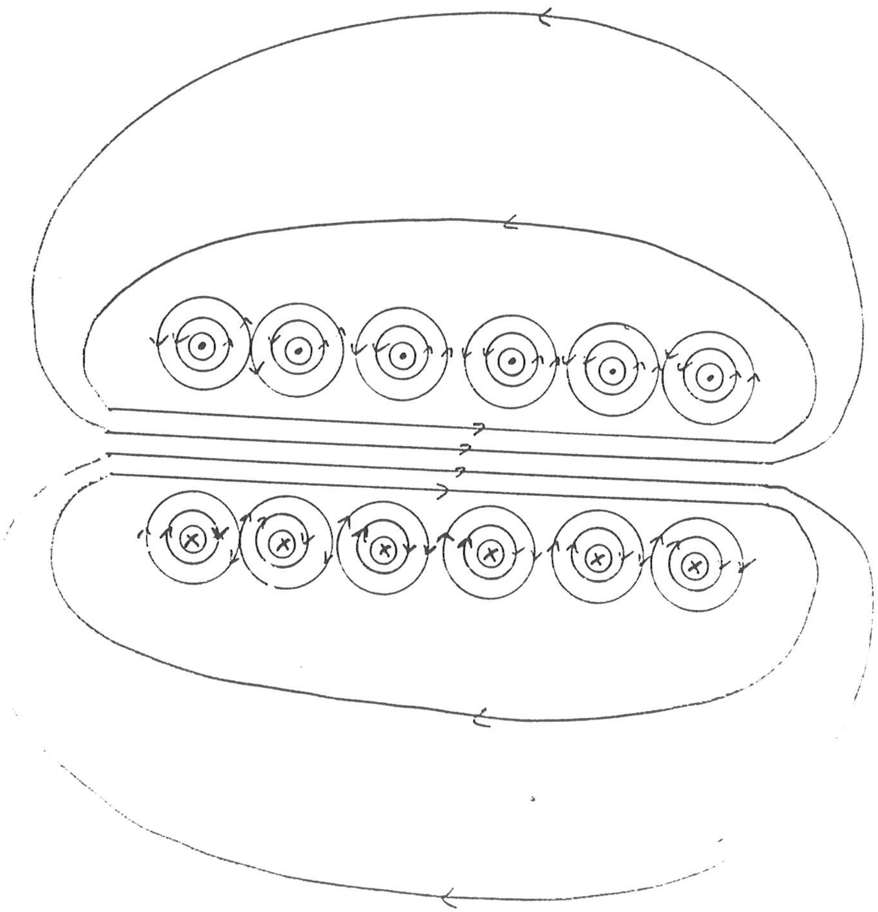
$B = \frac{\mu_0}{4\pi} \frac{i}{d} \int_{\varphi_1}^{\varphi_2} \sin\theta\, d\theta$

$B = \frac{\mu_0}{4\pi} \frac{i}{d} \left[\cos\varphi_1 - \cos\varphi_2\right]$

Check: Infinite wire $\varphi_1 = 0 \quad \varphi = 180$

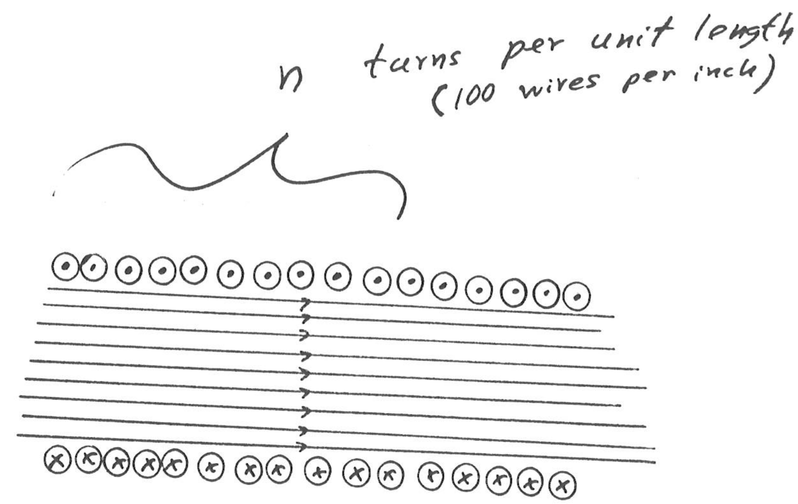
$B = \frac{\mu_0}{\cancel{4}\pi} \frac{i}{d} [2]$ ✓

The Solenoid



$\vec{B}$ field lines are <u>closed</u> loops.

The Solenoid



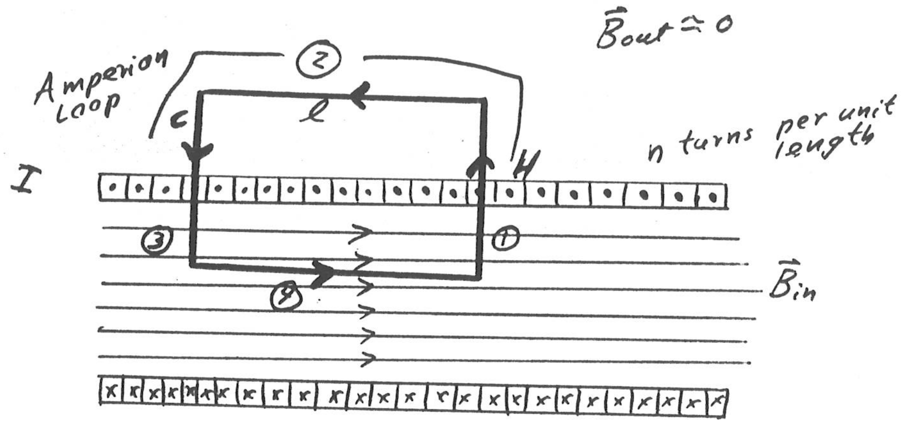
If the solenoid is very long compared to its radius and if the coils are closely spaced then:

$\vec{B}_{inside} \approx$ constant

$\vec{B}_{outside} \approx 0$

Well, not really, but the $\vec{B}$ field is <u>much</u> less dense outside.

Magnetic field inside a solenoid by Ampere's Law:



Ampere's Law: $\oint \vec{B}\cdot d\vec{\ell} = \mu_0 \, i_{enc}$

$$= \underbrace{\int_{1} \cancelto{0}{\vec{B}\cdot d\vec{\ell}}}_{\vec{B}\perp d\vec{\ell}} + \underbrace{\int_{2} \cancelto{0}{\vec{B}\cdot d\vec{\ell}}}_{B=0} + \underbrace{\int_{3} \cancelto{0}{\vec{B}\cdot d\vec{\ell}}}_{\vec{B}\perp d\vec{\ell}} + \int_{4} \vec{B}\cdot d\vec{\ell}$$

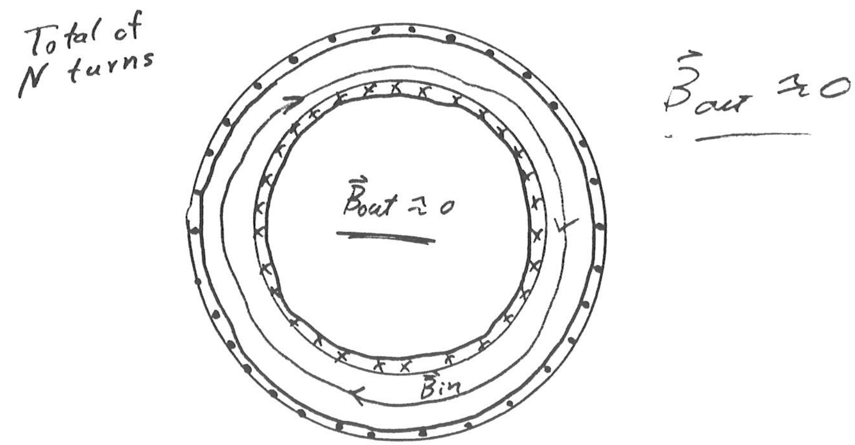
$$= \int_{4} \vec{B}\cdot d\vec{\ell} = |B| \int d\ell = B\ell$$

$$= \mu_0 \, i_{enc} = I \mu_0 \boxed{n\ell} \leftarrow \text{\# wires carrying } I$$

$$B\cancel{\ell} = \mu_0 I n \cancel{\ell} \qquad \boxed{B_{in} = \mu_0 I n}$$

The Toroid

Instead of making the solenoid infinitely long to get a very small $\vec{B}$ field outside, one can attach the open ends to each other to make a doughnut shape.

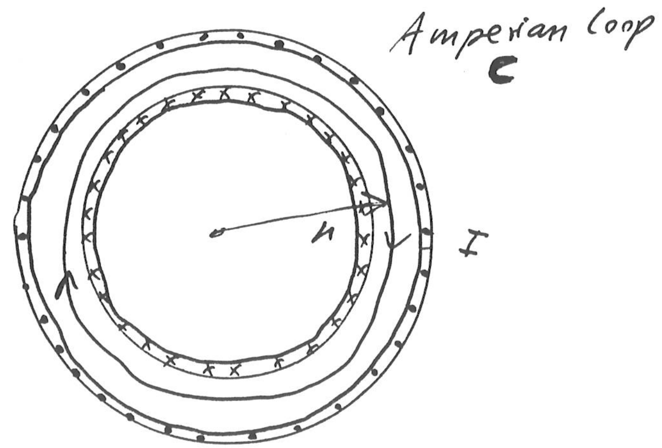


This time, $\vec{B}_{in} \neq$ constant.

The Toroid

Instead of making the solenoid infinitely long to get a very small $\vec{B}$ field outside, one can attach the open ends to each other to make a doughnut shape.

Total of N turns



Ampere's law: $\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 i_{enc}$

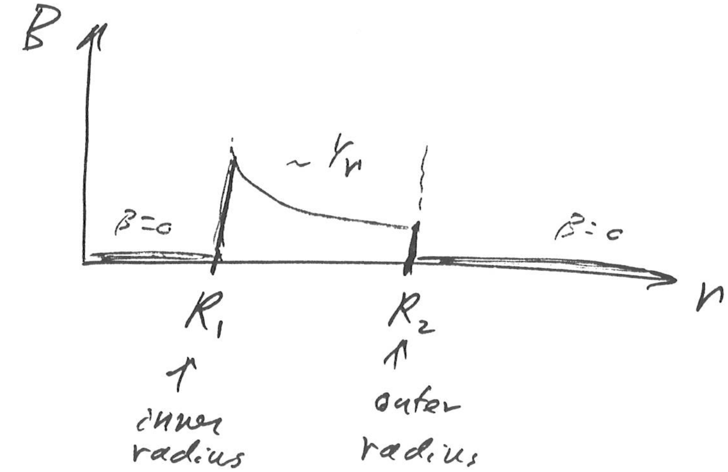
$\vec{B} \parallel d\vec{\ell}$ on Amperian Loop C

$\oint_C |B|\,|d\ell| = B \oint_C d\ell = B(2\pi r)$

$= \mu_0 i_{enc} = \mu_0 (N I)$

$$B_{inside} = \frac{\mu_0 N I}{2\pi r} = \boxed{\frac{\mu_0}{4\pi}\,\frac{2NI}{r}}$$

$B_{outside} = 0$



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n = wires per unit length

$= \dfrac{N}{2\pi r}$

$$B_{toroid} = \frac{\mu_0}{4\pi}\,\frac{2I}{r}(2\pi r n) = \mu_0 I n$$

$= B_{solenoid}$