Scale Functions $\{h_1, h_2, h_3\}$

Cartesian Coordinates $\{x, y, z\}$

unit vectors $\{\hat{i}, \hat{j}, \hat{k}\} = \{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$

$= \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

$h_1 = 1 \quad h_2 = 1 \quad h_3 = 1$

line element

Cartesian $\vec{ds} = dx\,\hat{e}_x + dy\,\hat{e}_y + dz\,\hat{e}_z$

$\vec{ds} \cdot \vec{ds} = |\vec{ds}|^2 = ds^2 = 1dx^2 + 1dy^2 + 1dz^2$

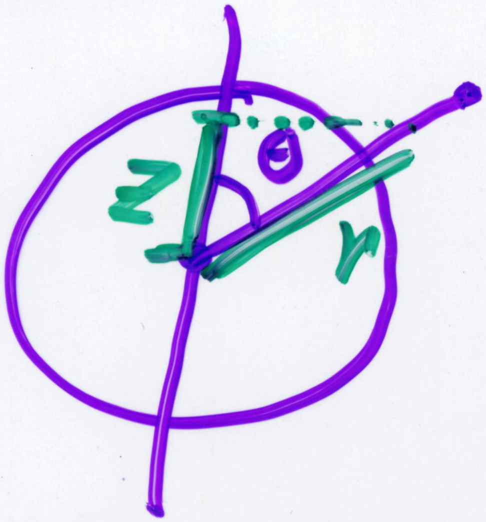
$= h_x^2 dx^2 + h_y^2 dy^2 + h_z^3 dz^2$

$h_x = 1$

$h_y = 1$

$h_z = 1$

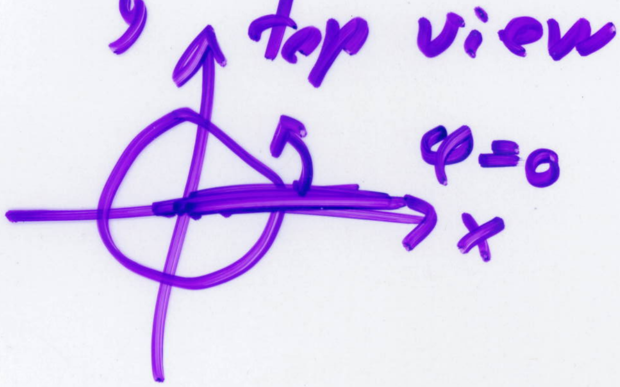
Spherical Polar Coords



$\theta = 0$ North pole

$\theta = \frac{\pi}{2}$ Equator

$\theta = \pi$ South pole



$$x = r \sin\theta \cos\varphi$$

$$y = r \sin\theta \sin\varphi$$

$$z = r \cos\theta$$

$$dx = dr \sin\theta \cos\varphi + r \cos\theta \, d\theta \cos\varphi - r \sin\theta \sin\varphi \, d\varphi$$

$$dy = dr \sin\theta \sin\varphi + r \cos\theta \, d\theta \sin\varphi + r \sin\theta \cos\varphi \, d\varphi$$

$$dz = dr \cos\theta - r \sin\theta \, d\theta$$

$$ds^2 = 1dx^2 + 1dy^2 + 1dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi$$

$$ds^2 = h_x^2 dx^2 + h_y^2 dy^2 + h_z^2 dz^2$$

$$ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\varphi^2 d\varphi^2$$

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scale factors Spherical polar

$h_r = 1$ $\quad$ $h_\theta = r$ $\quad$ $h_\varphi = r \sin\theta$

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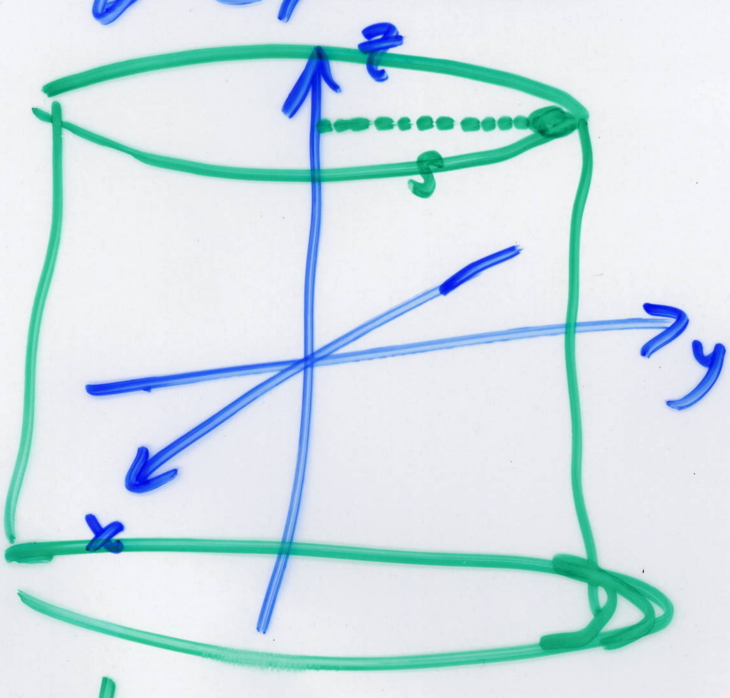
① Cylindrical Polar Coords

$\{s, \varphi, z\}$



$x = s \cos\varphi$

$y = s \sin\varphi$

$z = z$

$h_s = 1$ $\quad$ $h_\varphi = s$ $\quad$ $h_z = 1$

$$\iiint \cdots dV \qquad \int_x \int_y \int_z \cdots \underline{dx\,dy\,dz}$$

Cartesian $dV = dx\,dy\,dz$

Spherical $dV = dr\; r\,d\theta\; r\sin\theta\,d\varphi$

$$= r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Cylindrical $dV = s\,ds\,d\phi\,dz$

General:

$$dV = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$$

Gradient $\vec{\nabla}\Phi(\vec{r})$

Cartesian $\vec{\nabla}\Phi(x, y, z)$

$$= \hat{e}_x \frac{\partial \Phi}{\partial x} + \hat{e}_y \frac{\partial \Phi}{\partial y} + \hat{e}_z \frac{\partial \Phi}{\partial z}$$

Spherical $\vec{\nabla}\Phi(r, \theta, \varphi)$

$$= \hat{e}_r \frac{\partial \Phi}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \hat{e}_\varphi \frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \varphi}$$

General

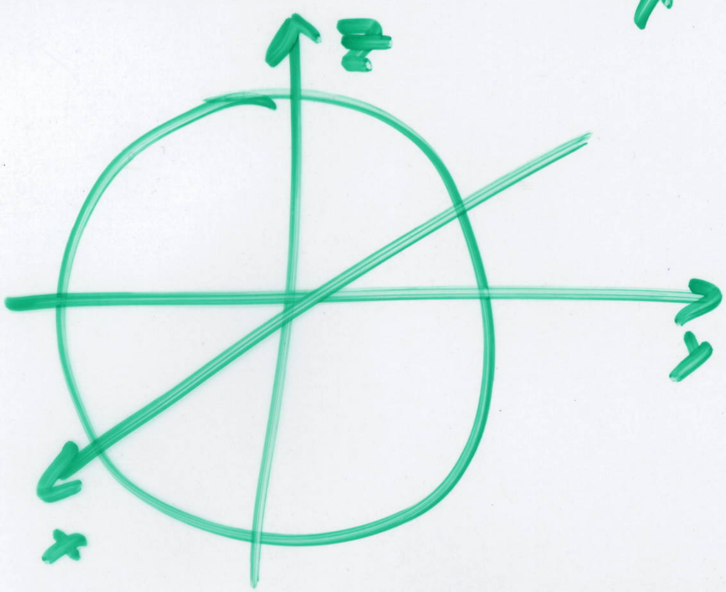
scale factors $\{h_1, h_2, h_3\}$

coordinates $\{q_1, q_2, q_3\}$

$\vec{\nabla}\Phi(q_1, q_2, q_3) =$

$$\hat{e}_1 \frac{1}{h_1}\frac{\partial \Phi}{\partial q_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial \Phi}{\partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial \Phi}{\partial q_3}$$

What are $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\varphi$



$\hat{e}_r$ = up

$\hat{e}_\theta$ = south

$\hat{e}_\varphi$ = east

Divergence $\vec{\nabla}\cdot\vec{V}(\vec{r})$

Cartesian:

$$\vec{\nabla}\cdot\vec{V}(x,y,z) = \frac{\partial V_x(x,y,z)}{\partial x} + \frac{\partial V_y(x,y,z)}{\partial y} + \frac{\partial V_z(x,y,z)}{\partial z}$$

Spherical

$$\vec{V}(\vec{r}) = \hat{e}_r V_r(r,\theta,\varphi) + \hat{e}_\theta V_\theta(r,\theta,\varphi) + \hat{e}_\varphi V_\varphi(r,\theta,\varphi)$$

$$\vec{\nabla}\cdot\vec{V}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 V_r\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\, V_\theta\right] + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left[V_\varphi\right]$$

Spherical

$h_1 = 1$

$h_2 = r$

$h_3 = r\sin\theta$

In general $\vec{V}(q_1, q_2, q_3)$

$$\vec{\nabla}\cdot\vec{V} \equiv \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(h_1 V_2 h_3) + \frac{\partial}{\partial q_3}(h_1 h_2 V_3)\right]$$

Example: point charge (monopole) at the origin, electrostatic potential = voltage

Cartesian

$$\Phi(\vec{r}) = \Phi(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2+z^2}}$$

Spherical

$$\Phi(\vec{r}) = \Phi(r, \theta, \varphi) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

Electric field $\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r})$

Cartesian

$$\vec{E}(\vec{r}) = \hat{e}_x E_x(x,y,z) + \hat{e}_y E_y(x,y,z) + \hat{e}_z E_z(x,y,z)$$

$$E_x(x,y,z) = -\frac{\partial}{\partial x}\left[\frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2+z^2}}\right]$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q\,x}{\sqrt{x^2+y^2+z^2}}$$

$E_y = \ldots$

$E_z = \ldots$

$\Phi(r, \cancel{\theta}, \cancel{\varphi}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$

Electric Field $\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r})$

$$\vec{E}(\vec{r}) = \hat{e}_r E_r(r,\theta,\varphi)$$
$$+ \cancel{\hat{e}_\theta E_\theta(r,\theta,\varphi)}$$
$$+ \cancel{\hat{e}_\varphi E_\varphi(r,\theta,\varphi)}$$

$$\vec{E}(\vec{r}) = -\hat{e}_r \frac{\partial}{\partial r}\left[\frac{1}{4\pi\epsilon_0}\frac{q}{r}\right] = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{e}_r$$
$$+ 0\,\hat{e}_\theta + 0\,\hat{e}_\varphi$$