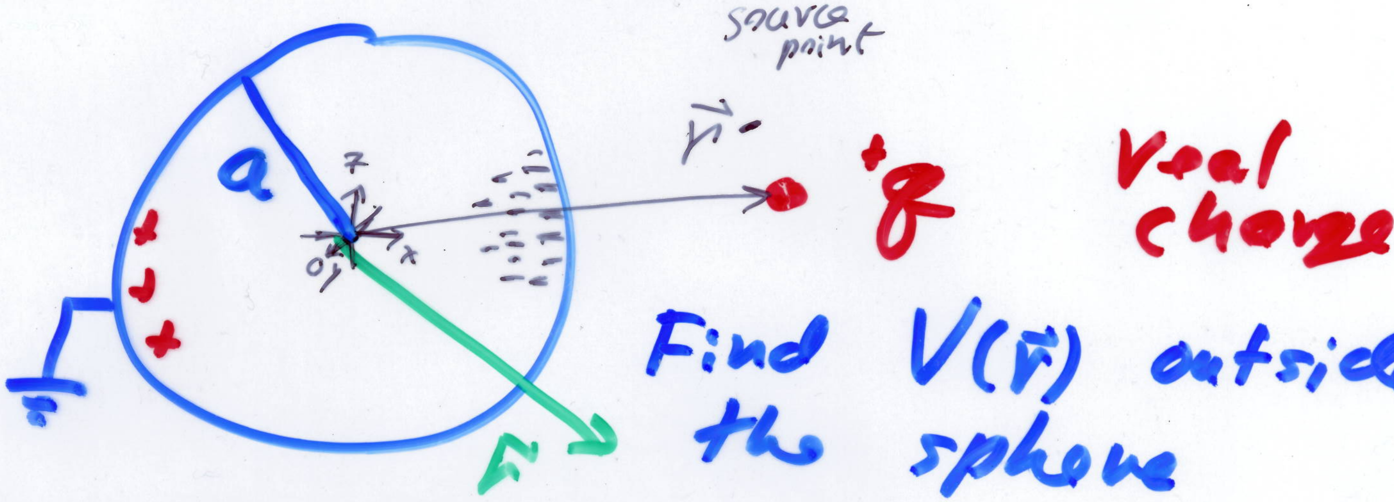
Method of Images

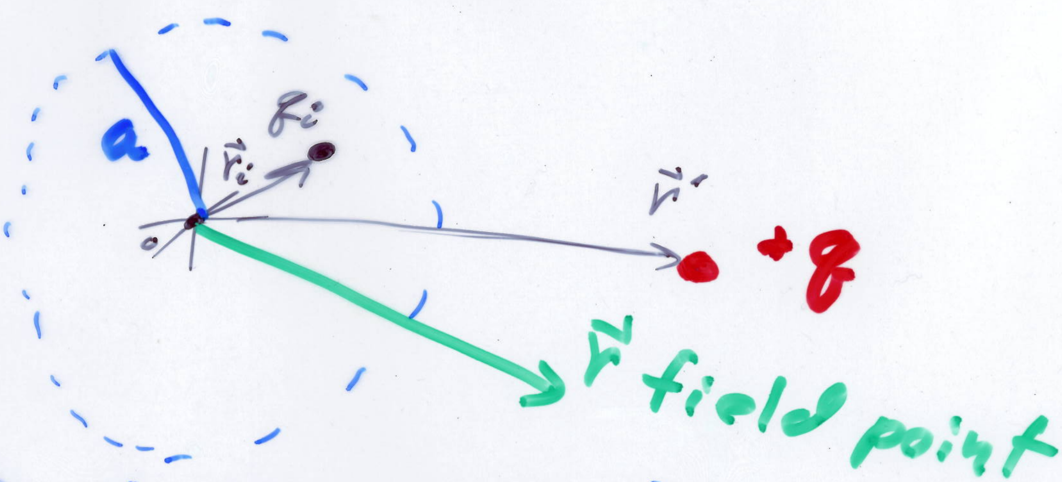
↑ Find the potential (voltage)

Real Problem



Find $V(\vec{r})$ outside the sphere

Image Problem



Find q_i and $\vec{r}_i$ such that $V=0$ on the sphere $|\vec{r}| = a$

$$V(\vec{r}) = \frac{kq}{|\vec{r}-\vec{r}'|} + \frac{kq_i}{|\vec{r}-\vec{r}_i|}$$

The boundary condition $V(\infty) \to 0$ is automatically satisfied.

On the surface of the sphere $|\vec{r}| = a$ $V(\vec{r})$ must be zero.

On the sphere, call the field point $\vec{a}$.

$$V_{\substack{\text{on}\\ \text{sphere}}} \equiv 0 = \frac{kq}{|\vec{a}-\vec{r}'|} + \frac{kq_i}{|\vec{a}-\vec{r}_i|}$$

q_i must have sign opposite to q

$$q_i = -\lambda q \quad \text{where } \lambda > 0$$

$$0 = \frac{\cancel{q}}{|\vec{a}-\vec{r}'|} - \frac{\lambda \cancel{q}}{|\vec{a}-\vec{r}_i|}$$

$$|\vec{a}-\vec{r}\,'| = \frac{1}{\lambda}|\vec{a}-\vec{r}_i|$$ square both sides

$$(\vec{a}-\vec{r}\,')\cdot(\vec{a}-\vec{r}\,') = \frac{1}{\lambda^2}(\vec{a}-\vec{r}_i)\cdot(\vec{a}-\vec{r}_i)$$

$$a^2 + (r')^2 - 2\vec{a}\cdot\vec{r}\,' = \frac{1}{\lambda^2}\left[a^2 + (r_i) - 2\vec{a} - \vec{r}_i\right]$$

$$\lambda^2\left[a^2 + (r')^2 - 2\vec{a}\cdot\vec{r}\,'\right] = a^2 + (r_i)^2 - 2\vec{a}\cdot\vec{r}_i$$

$$\underline{a^2(1-\lambda^2)} - \underline{2\vec{a}\cdot(\vec{r}_i - \lambda^2\vec{r}\,')} + \underline{(r_i)^2} - \underline{\lambda^2(r')^2} = 0$$

since this must hold for <u>any</u> direction of $\vec{a}$ (any θ, any φ) the vector

$\Rightarrow (\vec{r}_i - \lambda^2\vec{r}\,')$ must be zero.

Now know $\underline{\vec{r}_i = \lambda^2\vec{r}\,'}$

that is $\vec{r}_i$ and $\vec{r}\,'$ are parallel, or q_i is located between origin and q.

orange underlined terms

$$\Rightarrow a^2(1-\lambda^2) + \lambda^4 (r')^2 - \lambda^2 (r')^2 = 0$$

reject: image is outside sphere

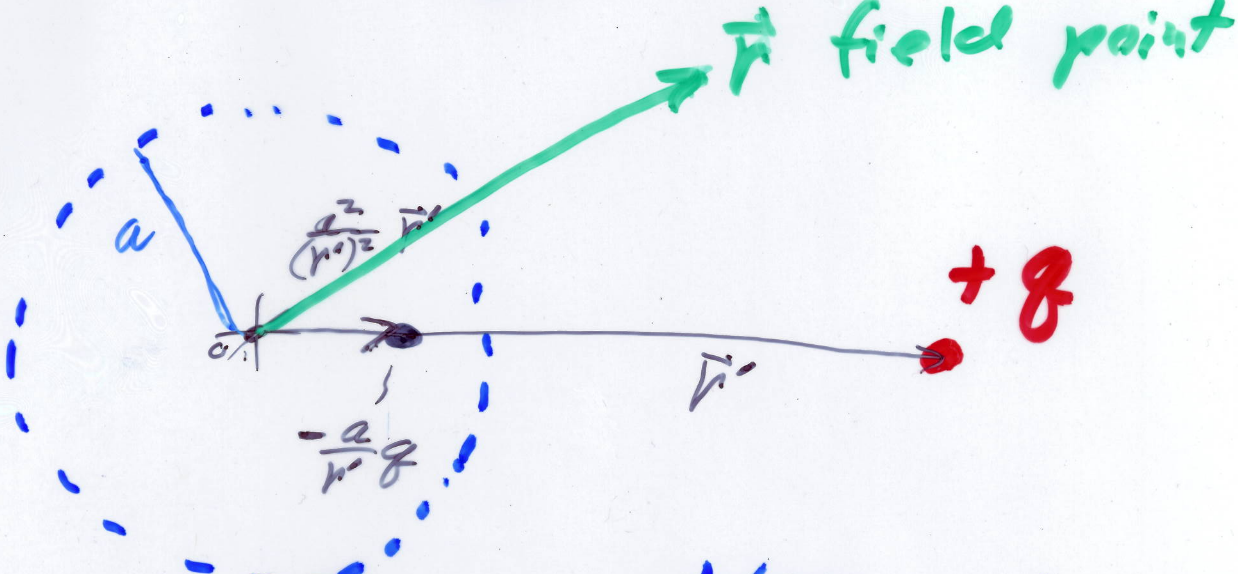
4 solutions: $\lambda = \boxed{+1}, \boxed{-1}, \frac{a}{r'}, \boxed{-\frac{a}{r'}}$

reject since $\lambda > 0$.

physical solution $\lambda = \frac{a}{r'}$

$$q_i = -\lambda q = -\frac{a}{r'} q$$

$$\vec{r}_i = \lambda^2 \vec{r}' = \frac{a^2}{(r')^2} \vec{r}'$$



Now easy to find $V(\vec{r})$ outside:

$$V(\vec{r}) = \frac{kq}{|\vec{r}-\vec{r}'|} - \frac{k\frac{a}{r'}q}{\left|\vec{r}-\frac{a^2}{(r')^2}\vec{r}'\right|}$$

Valid for $|\vec{r}| \geq a$ (outside)

What else can we do?

Electric field outside $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$

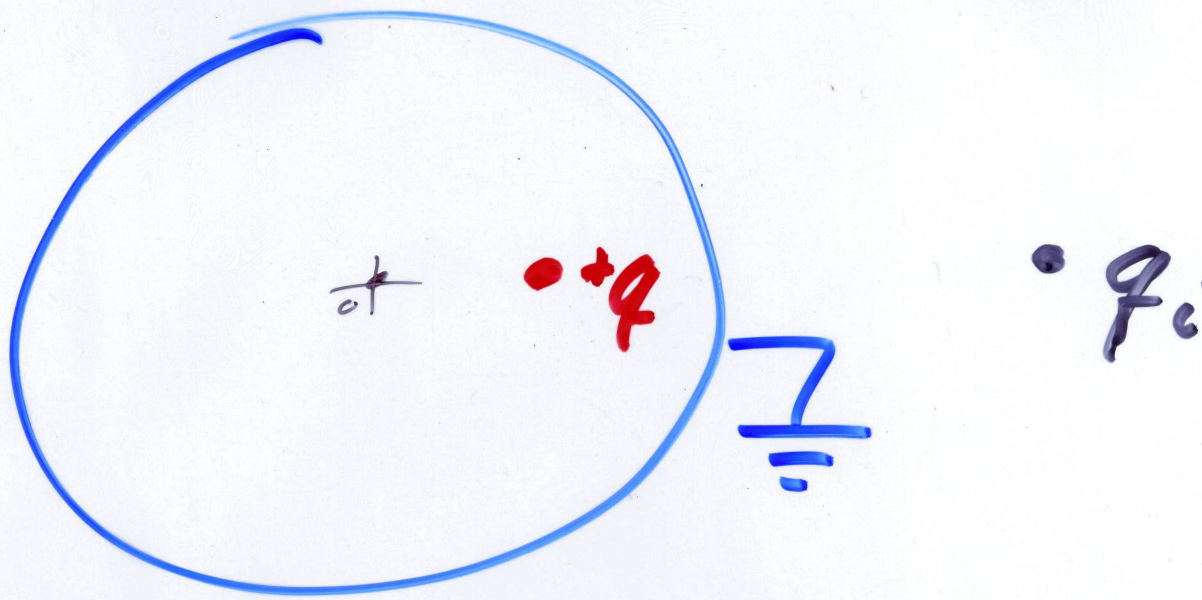
Surface charge density on sphere

$$\sigma(a,\theta,\varphi) = -\epsilon_0 \hat{r}\cdot\vec{E}(a,\theta,\varphi)$$

Total charge on sphere.

$$Q_{Total} = \oiint_S \sigma(a,\theta,\varphi)\, a^2 \sin\theta\, d\theta\, d\varphi$$

The same $V(\vec{r})$ also solves the <u>interior</u> problem.



Find $V(\vec{r})$ inside the sphere.