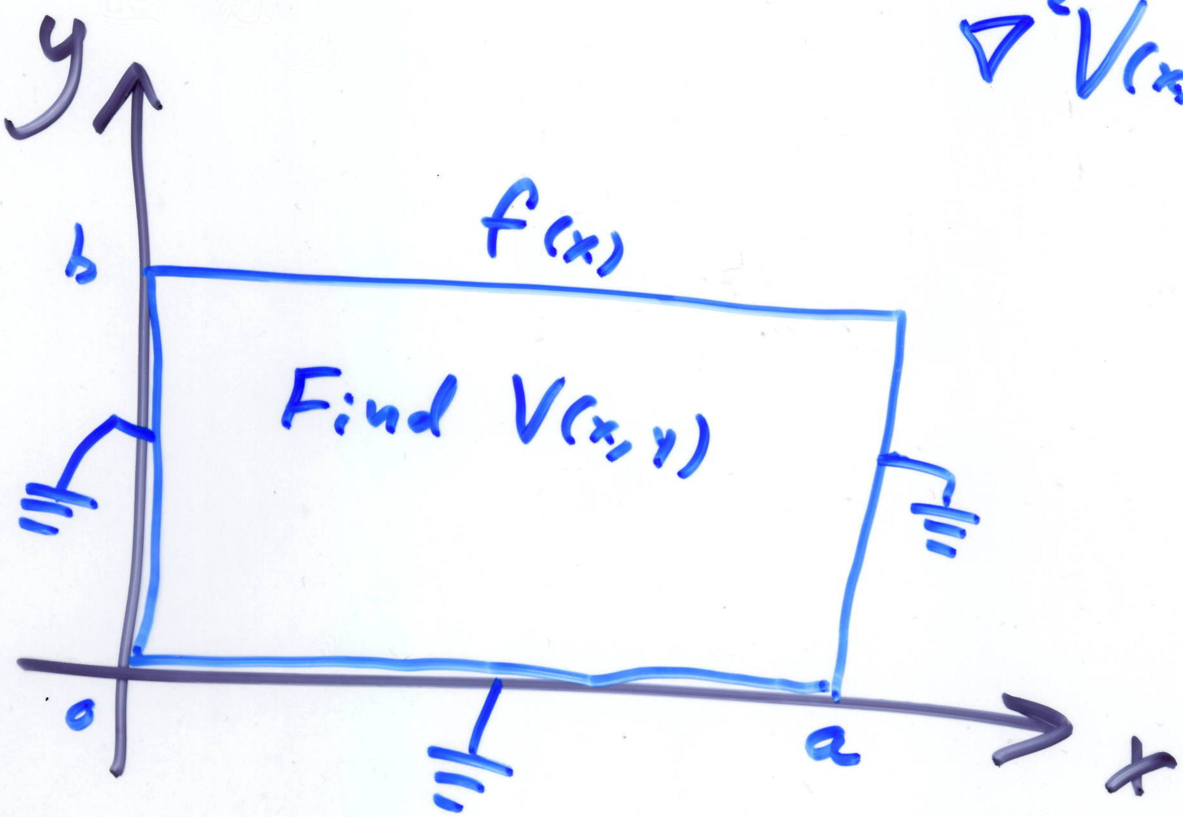
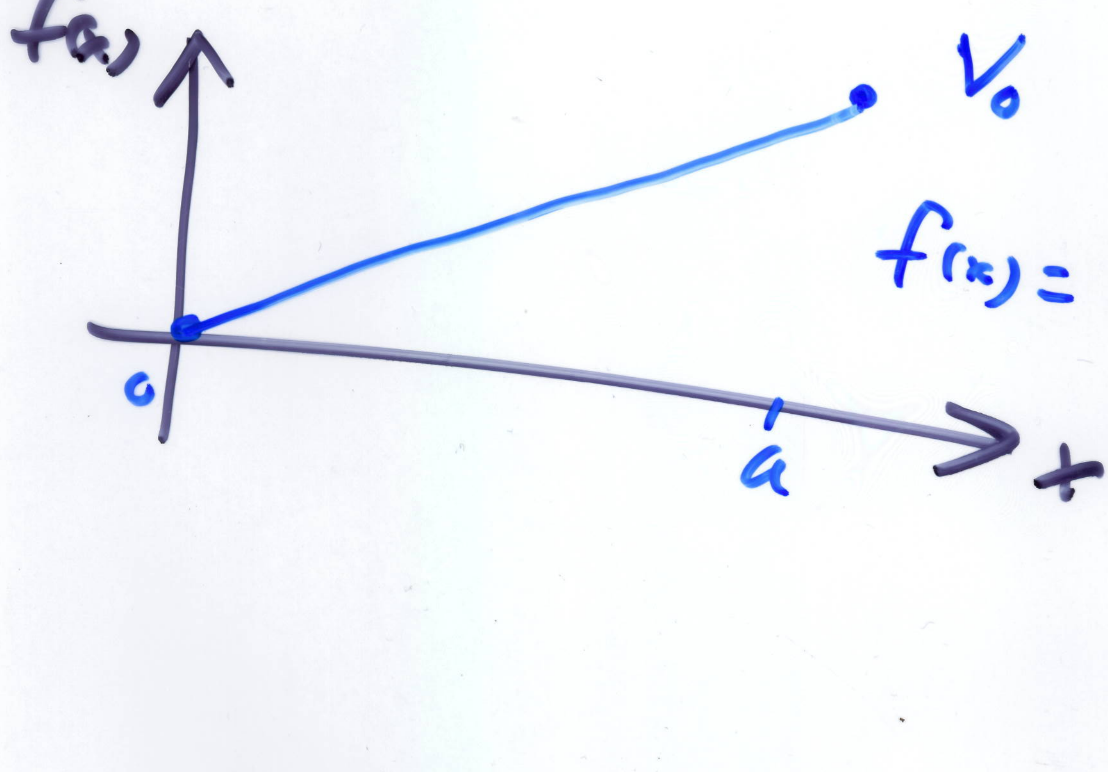
New Homework

$$\nabla^2 V(x,y) = 0$$

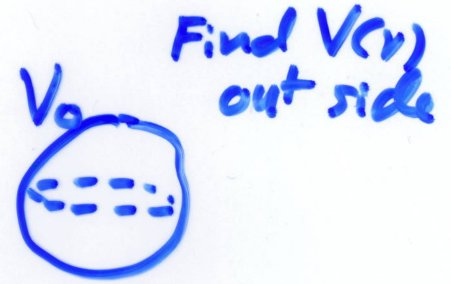


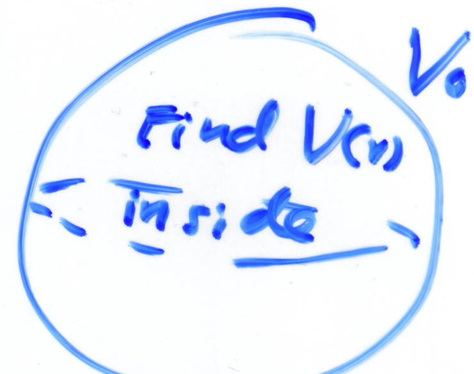


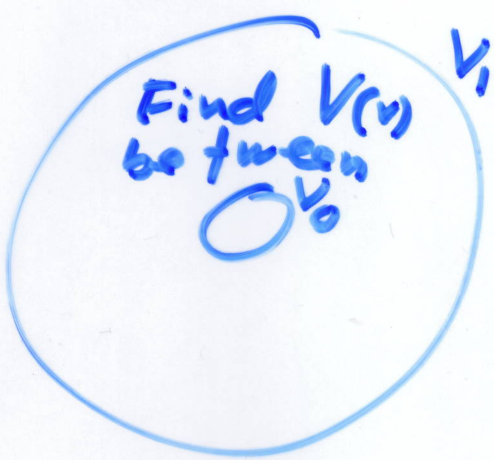
$$f(x) = \frac{V_0}{a} x$$

Separation of Variables in Spherical Polar Coordinates.

I a) B.C. depend on one Coordinate radial coordinate r.







Solution $V(\vec{r})$ can only depend on r, not θ, not φ

$$\nabla^2 V(\vec{r}) = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr} V(r)\right] = 0$$

$$\frac{d}{dr}\left[r^2 \frac{d}{dr} V(r)\right] = 0$$

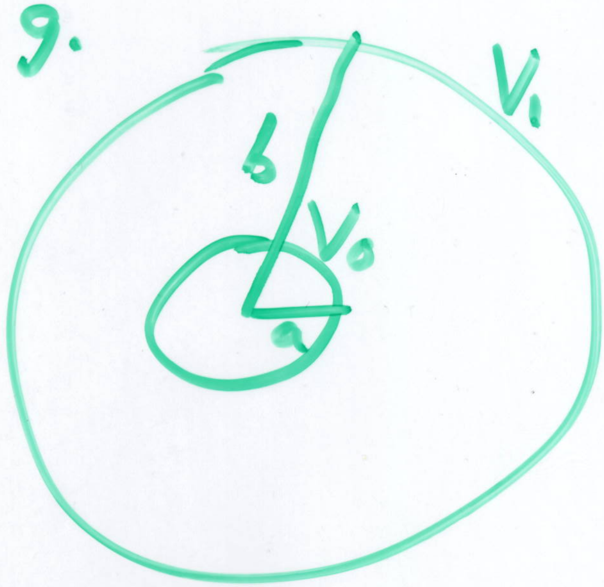
$$\left[r^2 \frac{d}{dr} V(r)\right] = \text{constant} = -C_1$$

$$\frac{dV(r)}{dr} = -\frac{c_1}{r^2}$$

$$\boxed{V(r) = \frac{c_1}{r} + C_2}$$

Must set $c_1 = 0$ if the origin $r = 0$ is included.

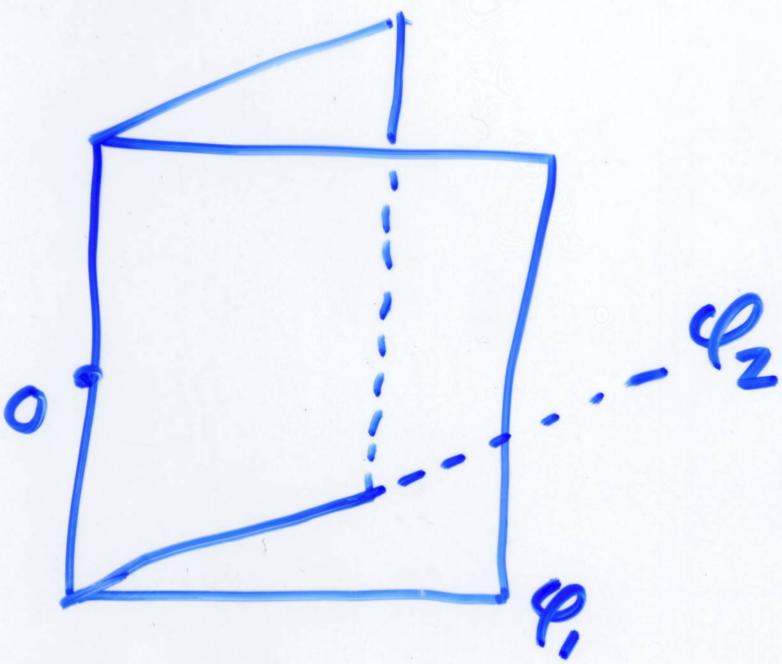
E.g.



Find $V(r)$ between the shells.

Find c_1 and c_2.

I b) B.C. depend only on φ.



$$\nabla^2 V(\vec{r}) = 0$$

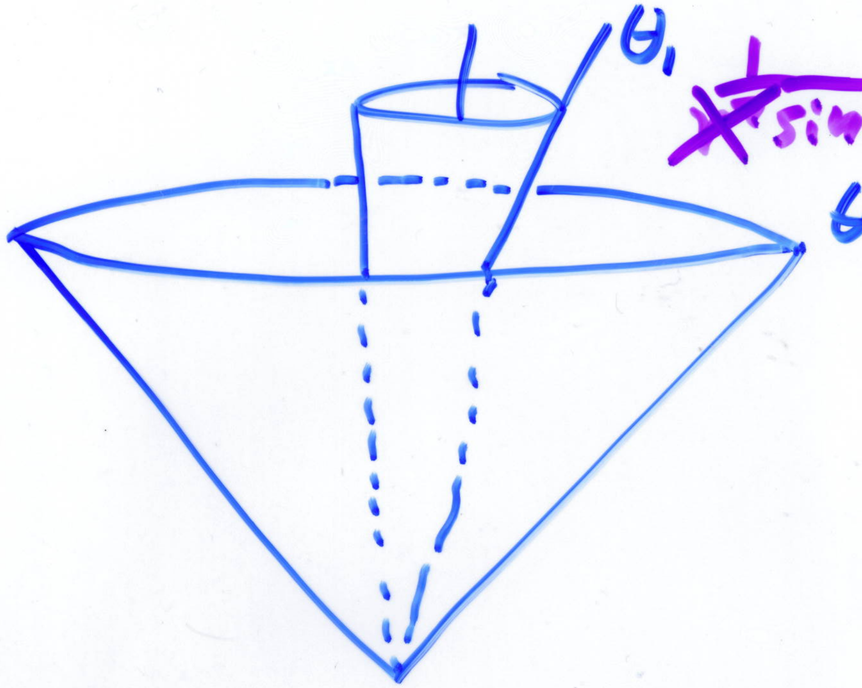
$$\frac{d^2}{d\varphi^2} V(\varphi) = 0$$

$$V(\varphi) = c_1 \varphi + c_2$$

I c) B.C. depend only on θ

$$\nabla^2 V(\vec{r}) = 0$$

$$\frac{1}{\cancel{r^2}\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dV}{d\theta}\right]=0$$



II) B.C. depend on 2 coordinates.
r, θ not φ. (azimuthal symmetry).

$$\nabla^2 V(r,\theta,\varphi) = 0$$

$$= \frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial V}{\partial\theta}\right] +$$

$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Ansatz: $V(r,\theta) = R(r)\cdot T(\theta)$

$$0 = \frac{1}{R(r)} \frac{d}{dr}\left[r^2 \frac{dR(r)}{dr}\right] + \frac{1}{T(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{dT(\theta)}{d\theta}\right]$$

$$\Rightarrow \frac{1}{R(r)} \frac{d}{dr}\left[r^2 \frac{dR(r)}{dr}\right] = \text{separation constant} = l(l+1)$$

$$\frac{1}{T(\theta)\sin\theta} \frac{d}{d\theta}\left[\sin\theta \frac{dT(\theta)}{d\theta}\right] = -\text{sep. const.} = -l(l+1)$$

$$\frac{d}{dr}\left[r^2 \frac{dR(r)}{dr}\right] = l(l+1)\,R(r)$$

solutions: $R(r) = A\,r^l + B\,\frac{1}{r^{l+1}}$

$$\frac{d}{d\theta}\left[\sin\theta \frac{dT(\theta)}{d\theta}\right] = -l(l+1)\sin\theta\, T(\theta)$$

solutions: $T(\theta) = C\,P_l(\cos\theta)$

$+ D\,Q_l(\cos\theta)$

Q_l badly behaved on the axis.

If North Pole ($\theta = 0$) or South Pole ($\theta = \pi$) are included then you must set $D = 0$, because $Q_\ell(\cos\theta)$ blows up there.

$P_\ell(\cos\theta)$ are called Legendre Polynomials.

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$$

$$P_3(\cos\theta) = \frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta$$

All normalized (arbitrarily) such that at $\theta = 0$ (North pole) where $\cos\theta = 1$, we have $P_\ell(1) = 1$.

Legendre Polynomials are orthogonal (like sine + cosines).

$$\int_{\theta=0}^{\pi} P_\ell(\cos\theta)\cdot P_n(\cos\theta)\,\sin\theta\,d\theta = \left\{ \frac{2}{2\ell+1}\,\delta_{\ell n} \right.$$

Price to pay for normalizing $P_\ell(1) = 1$.

General Solution (with azimuthal symmetry) and Polar axis included (no Q_ℓ's)

$$V(r,\theta) = R(r)\cdot T(\theta) =$$

$$= \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

E.g.

B.C. $V(R,\theta) = f(\theta)$

Must set $B_\ell = 0$ since origin $r=0$ is included.

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

guaranteed to solve $\nabla^2 V(\vec{r}) = 0$

now satisfy the B.C.

$$V(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = f(\theta)$$

Use orthogonality equation for $P_\ell(\cos\theta)$ to solve for A_ℓ

Multiply both sides by

$P_n(\cos\theta)\sin\theta$ and integrate

$$\int_0^\pi d\theta$$

$$\sum_{l=0}^{\infty} A_l R^l \overbrace{\boxed{\int_{\theta=0}^{\pi} P_l(\cos\theta)\, P_n(\cos\theta) \sin\theta \, d\theta}}^{\frac{2}{2l+1}\delta_{nl}}$$

Fourier's trick

$$= \int_{\theta=0}^{\pi} f(\theta)\, P_n(\cos\theta) \sin\theta \, d\theta$$

$$A_n R^n \frac{2}{2n+1} = \int_{\theta=0}^{\pi} f(\theta)\, P_n(\cos\theta) \sin\theta \, d\theta$$

$$A_n = \ldots$$