Electromagnetism

Foundations & Maxwell's Equations

Topics covered: Laplace's equation, electric & magnetic monopoles, Maxwell's equations with hypothetical monopole terms, Coulomb's law, Lorentz force, vector potential, and magnetic dipoles.

1 Minimal Surfaces vs. Electrostatic Potential

A soap film (or rubber sheet) in equilibrium satisfies a second-order **nonlinear** PDE that minimizes its area. If z(x, y) is the height of the surface, the governing equation is:

$$\frac{\partial}{\partial x}\left[\frac{\partial z/\partial x}{\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}} \right] + \frac{\partial}{\partial y}\left[\frac{\partial z/\partial y}{\sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}} \right] = 0$$

By contrast, the electrostatic potential V(x, y) satisfies **Laplace's equation**, a second-order **linear** PDE:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) V(x, y) = 0 \quad \Rightarrow \quad \nabla^2 V(x, y) = 0$$

The key distinction: the soap-film equation is nonlinear (the denominator depends on the gradient of z), while Laplace's equation is linear, enabling superposition of solutions.

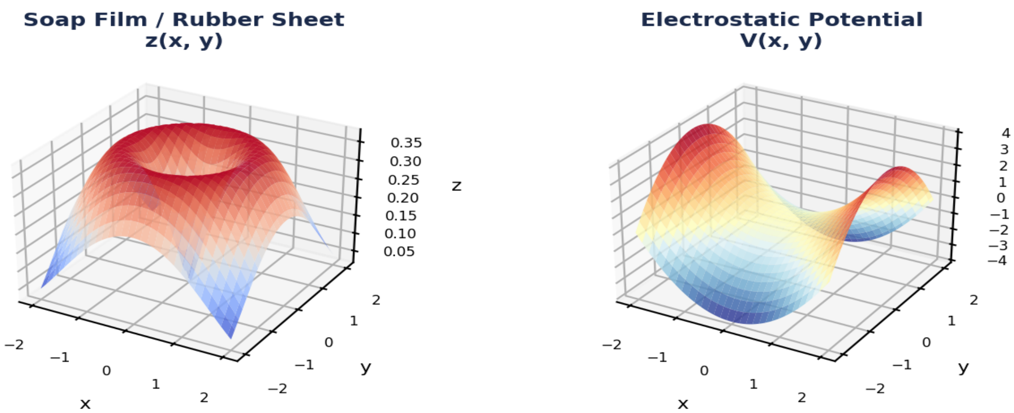

Figure 1 — Comparison of a soap film surface z(x, y) and an electrostatic saddle-point potential V(x, y).

2 Electric & Magnetic Monopoles

An **electric monopole** (point charge q) produces radial electric field lines: outward for $q > 0$, inward for $q < 0$. A hypothetical **magnetic monopole** (magnetic charge g) would produce radial magnetic field lines in the same pattern, with "north" ($g > 0$) acting as a source and "south" ($g < 0$) as a sink.

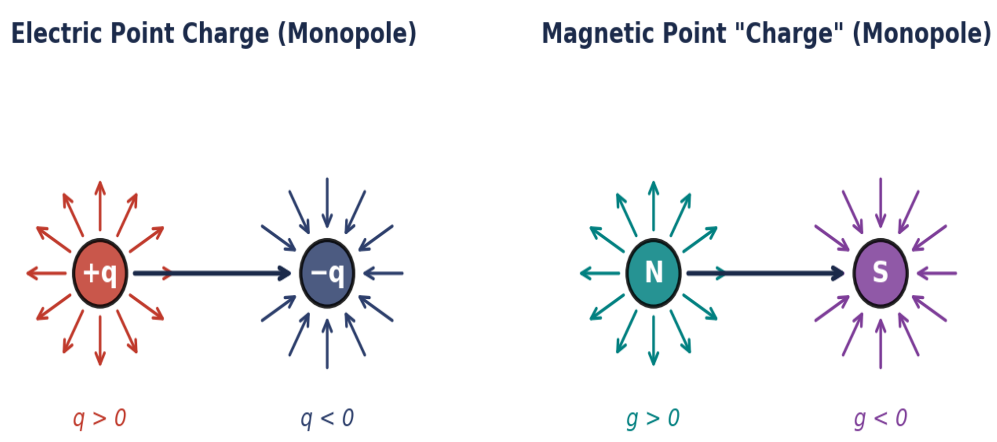

Figure 2 — Field lines for electric charges (left) and hypothetical magnetic monopoles (right).

3 Maxwell's Equations

Maxwell's four equations govern all of classical electrodynamics. In their standard differential form (Heaviside formulation), they are presented below. The **purple terms** show how the equations would be modified if magnetic monopoles were discovered.

Law	Standard Form	Monopole Term
Gauss's Law	$\nabla \cdot \mathbf{E}(r, t) = \rho_{enc} / \varepsilon_0$	—
Gauss's Law (B)	$\nabla \cdot \mathbf{B}(r, t) = 0$	$+ \mu_0 \rho_{mag}(r, t)$
Ampère–Maxwell	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t$	—
Faraday's Law	$\nabla \times \mathbf{E}(r, t) = -\partial \mathbf{B}/\partial t$	$+ \mu_0 \mathbf{J}_{mag}(r, t)$

*Note: In Ampère–Maxwell's law, the first term (μ_0 **J**) is the **electric current** contribution and the second term ($\mu_0\varepsilon_0\, \partial \mathbf{E}/\partial t$) is the **displacement current**, added by Maxwell.*

4 Units, Constants, & Fundamental Laws

Key relation:

$$\mu_0 \, \varepsilon_0 = 1 / c^2$$

SI Units:

Quantity	Symbol	SI Unit
Electric charge	q	C (coulombs)
Magnetic charge	g	A·m (amp-meters)
Electric field	**E**	V/m = N/C
Magnetic field	**B**	T (tesla)

Coulomb's Law — force on charge 1 due to charge 2:

$$\mathbf{F}_{12} = (1 / 4\pi\varepsilon_0) \cdot (q_1 \, q_2 / r_{12}^{\,2}) \cdot r\blacksquare_{12}$$

where $r\blacksquare_{12} = \mathbf{r}_{12} / |\mathbf{r}_{12}|$ is the unit vector from charge 2 to charge 1, and $r\blacksquare_{12} \cdot r\blacksquare_{12} = 1$.

Force Between Parallel Currents:

$$\mathbf{F}_{12} = (\mu_0 / 4\pi) \cdot (i_1 \, i_2 \cos\theta / r_{12}) \cdot r\blacksquare_{12}$$

5 Vector Potential & Displacement Field

The **magnetic vector potential A** is defined such that:

$$\nabla \times \mathbf{A} = \mathbf{B}$$

The **displacement field D** accounts for bound charges in media, while **E** is the electric field.

6 The Lorentz Force

The force on an electric charge q moving with velocity **v** in external electric field **E** and magnetic field **B** is:

$$\mathbf{F}_q = q\,(\,\mathbf{E} + \mathbf{v} \times \mathbf{B}\,)$$

Here **E** is the electric field produced by all other charges in the universe, and **B** is the magnetic field from all other currents in the universe.

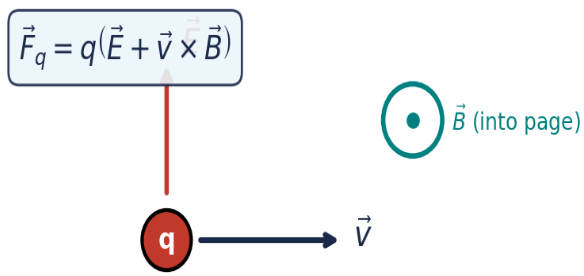

Figure 3 — A charge q moving with velocity v through E and B fields.

Force on a Hypothetical Magnetic Monopole:

$$\mathbf{F}_g = g\,(\,\mathbf{B} - \mathbf{v} \times \mathbf{E} / c^2\,)$$

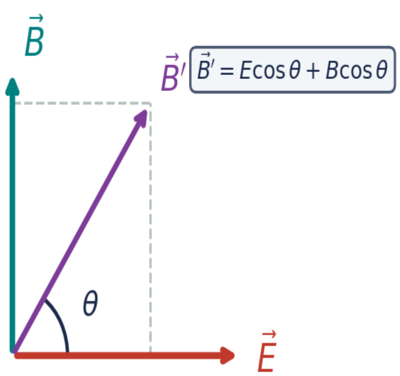

Figure 4 — Decomposition of E and B field components at angle θ.

7 Magnetic Forces & Work

A key result from Griffiths: **"Magnetic forces do no work."**

More precisely: for an electric point charge, the power delivered is $P = dW/dt = \mathbf{F} \cdot \mathbf{v}$. Expanding with the Lorentz force:

$$P = q\,\mathbf{E} \cdot \mathbf{v} + q\,(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}$$

Since $\mathbf{a} \times \mathbf{a} = 0$ for any vector, the cross product ($\mathbf{v} \times \mathbf{B}$) is always perpendicular to $\mathbf{v}$, so $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$. Therefore the magnetic force contributes zero power — it does no work on electric monopoles (point charges).

8 Magnetic Dipoles

A **magnetic dipole** (also called a point dipole or mathematical dipole) has potential energy in an external magnetic field given by:

$$U = -\mu \cdot \mathbf{B}$$

The dipole moment μ for two charges $\pm q$ separated by distance d is:

$$\mu = q \cdot d$$

A dipole consists of equal and opposite charges (+q and –q) separated by a displacement *d*. The dipole moment vector μ points from –q to +q.

In a uniform external field **B**, the dipole experiences a torque $\tau = \mu \times \mathbf{B}$ that tends to align μ with **B**. The energy is minimized when μ is parallel to **B**.

Figure 5 — Magnetic dipole: charges ±q separated by distance d.