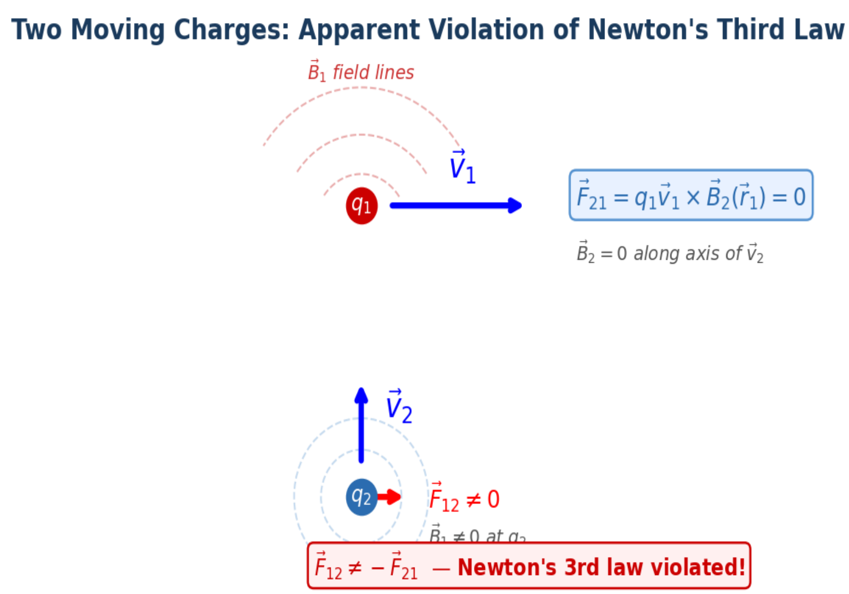
PHYS 4392 - Lecture Notes

Lecture 4 | Southern Methodist University

1 Newton's Third Law and Electromagnetic Momentum

Consider two point charges in motion: charge q_1 moves to the right with velocity v_1, and charge q_2 (located below q_1) moves upward with velocity v_2. We analyze the magnetic forces each exerts on the other.



1.1 Analyzing the Forces

Force on q_1 due to q_2: Charge q_2 moves upward, creating a magnetic field B_2. The magnetic field of a moving point charge is zero along the axis of its velocity. Since q_1 lies on the axis of v_2:

$$\vec{F}_{21} = q_1\,\vec{v}_1 \times \vec{B}_2(\vec{r}_1) = 0$$

Force on q_2 due to q_1: Charge q_1 moves to the right, creating B_1 that circulates around its velocity axis. At q_2 (below q_1), this field is *not* zero:

$$\vec{F}_{12} = q_2\,\vec{v}_2 \times \vec{B}_1(\vec{r}_2) \neq 0$$

1.2 The Apparent Violation

Newton's third law requires equal and opposite action-reaction pairs:

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = \vec{F}_{12} + \vec{F}_{21}$$

$$\text{If } \ \vec{p}_1 + \vec{p}_2 = \text{const.} \quad \Longrightarrow \quad \vec{F}_{12} = -\vec{F}_{21}$$

But F_{21} = 0 while F_{12} is nonzero. This *appears* to violate Newton's third law!

1.3 The Resolution: Field Momentum

The resolution is that we are using an incomplete definition of momentum. The electromagnetic field itself carries momentum with density:

$$\vec{g} = \mu_0 \epsilon_0 (\vec{E} \times \vec{B}) = \frac{1}{c^2}(\vec{E} \times \vec{B})$$

When we include the momentum stored in the electromagnetic field, total momentum is conserved:

$$\vec{p}_{\text{mech}} + \vec{p}_{\text{field}} = \text{constant}$$

Note: *Newton's third law in its strong form applies only to mechanical momentum. In electrodynamics, fields carry energy and momentum, so the conservation law must be generalized to include both. This connects to the Poynting vector and the electromagnetic stress-energy tensor.*

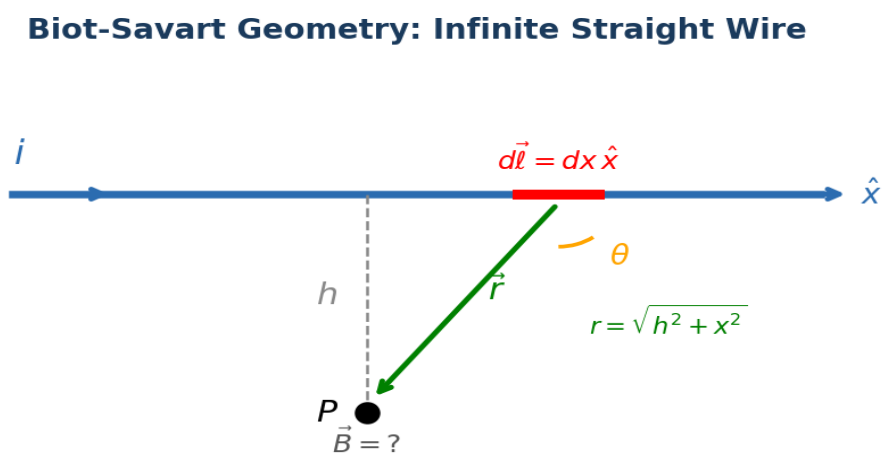
2 The Biot-Savart Law

The Biot-Savart law gives the magnetic field produced by a steady current. For a current-carrying wire, the differential contribution from an infinitesimal current element is:

$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \vec{r}}{r^3}$$

where **dl** is a vector element along the wire in the direction of current flow, **r** is the vector from the current element to the field point, and r-hat is the corresponding unit vector.

2.1 Application: Infinite Straight Wire



For an infinite straight wire along the x-axis carrying current i, the field at perpendicular distance h is found by integrating from x = -infinity to +infinity. The cross product magnitude is |dl x r-hat| = dl sin(θ), where sin(θ) = h/r:

$$|\vec{B}| = \frac{\mu_0\, i}{4\pi} \int_{-\infty}^{\infty} \frac{h\, dx}{(h^2 + x^2)^{3/2}}$$

$$= \frac{\mu_0\, i\, h}{4\pi} \left[\frac{x}{h^2\sqrt{h^2 + x^2}}\right]_{-\infty}^{\infty} = \frac{\mu_0\, i}{4\pi} \cdot \frac{2}{h}$$

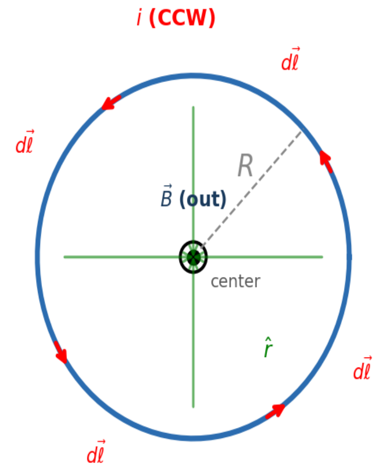
The direction is found by the right-hand rule. The final result:

$$\vec{B} = \frac{\mu_0\, i}{2\pi\, h}\, \hat{\phi}$$

This confirms ($\mu_0/4\pi$)(2/h) = $\mu_0/(2\pi h)$.

2.2 Application: Circular Current Loop (Field at Center)

Circular Current Loop: B at Center via Biot-Savart



For a circular loop of radius R carrying current i (CCW), we find B at the center. Key simplifications:

- Every element dl is at distance |r| = R from the center.
- dl is always perpendicular to r-hat (θ = 90 degrees, sinθ = 1).
- The arc length element is dl = R dϕ.

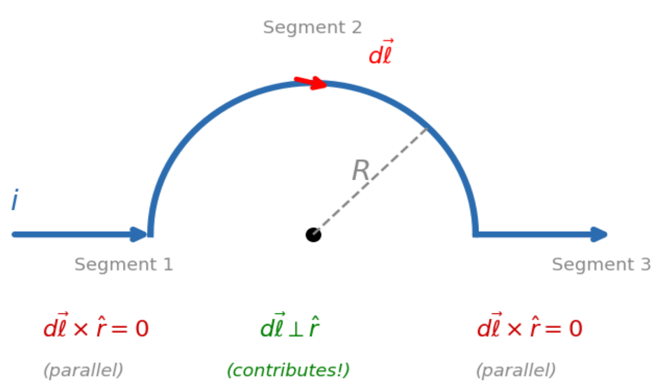
$$|\vec{B}| = \frac{\mu_0\, i}{4\pi} \int_0^{2\pi} \frac{R\, d\phi}{R^2} = \frac{\mu_0\, i}{4\pi\, R}\,(2\pi)$$

$$\vec{B}_{\text{center}} = \frac{\mu_0\, i}{2R}\,\hat{n}$$

where n-hat is the normal to the loop (out of the page for CCW current, by the right-hand rule).

2.3 Application: Partial Circular Arc

Partial Circle: Only the Curved Segment Contributes



For a wire with straight segments and a semicircular arc of radius R:

Straight segments: dl is parallel to r-hat, so dl x r-hat = 0. No contribution.

Semicircular arc: Integrate over $\phi = 0$ to π instead of 0 to 2π:

$$\vec{B} = \frac{\mu_0\, i}{4\pi\, R}\,(\pi) = \frac{\mu_0\, i}{4R}$$

General rule: For an arc subtending angle $\Delta\phi$ (radians):

$$B = \frac{\mu_0\, i}{4\pi\, R}\,\Delta\phi$$

3 From Biot-Savart to Ampere's Law

In mechanics, we can solve problems using F = ma directly or via conservation of energy. An analogous duality exists in E&M:

	Electrostatics	Magnetostatics
Direct calculation	Coulomb's law	Biot-Savart law
Symmetry method	Gauss's law	Ampere's law

3.1 The Laws Side by Side

Coulomb's Law	Biot-Savart Law
$\vec{F}_{12} = k_e \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12}$	$d\vec{B} = \frac{\mu_0}{4\pi} \frac{i\, d\vec{\ell} \times \hat{r}}{r^2}$
Gauss's Law	**Ampere's Law**
$\oint_S \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\epsilon_0}$	$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0\, i_{\text{enc}}$

3.2 Ampere's Law

Just as Gauss's law relates electric flux through a closed surface to enclosed charge, Ampere's law relates the circulation of B around a closed loop to the enclosed current:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0\, i_{\text{enc}}$$

Here, C is any closed curve (the *Amperian loop*), dl is a differential element along C, and i_{enc} is the total current threading any surface bounded by C. The sign convention follows the right-hand rule.

Ampere's law is most powerful when the current distribution has high symmetry, allowing B to be pulled out of the integral. Common applications include infinite straight wires, solenoids, and toroids.

4 Summary of Key Takeaways

1. For two moving point charges, magnetic forces are generally **not** equal and opposite. The electromagnetic field itself carries momentum, restoring total momentum conservation.

2. The Biot-Savart law gives B from a steady current: **dB = (μ_0/4π) i dl x r-hat / r^2**.

3. Infinite straight wire: **B = μ_0 i / (2πh)**, circling the wire per the right-hand rule.

4. Circular loop at the center: **B = μ_0 i / (2R)**. For a partial arc of angle Δϕ: **B = (μ_0 i / 4πR)** Δϕ.

5. Ampere's law, ■ **B · dl = μ_0 i_{enc}**, is the magnetic analog of Gauss's law and is the tool of choice when symmetry allows.