Electromagnetism Lecture Notes

April 1 — Ampère's Law, Solenoids, & Toroids

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1. Equivalence of Differential & Integral Forms

We want to show that the differential and integral forms of Ampère's law (magnetostatics) are equivalent. Starting from the differential form:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Integrate both sides over an open surface S:

$$\iint_S (\nabla \times \mathbf{B}) \cdot d\mathbf{A} = \iint_S \mu_0 \mathbf{J} \cdot d\mathbf{A}$$

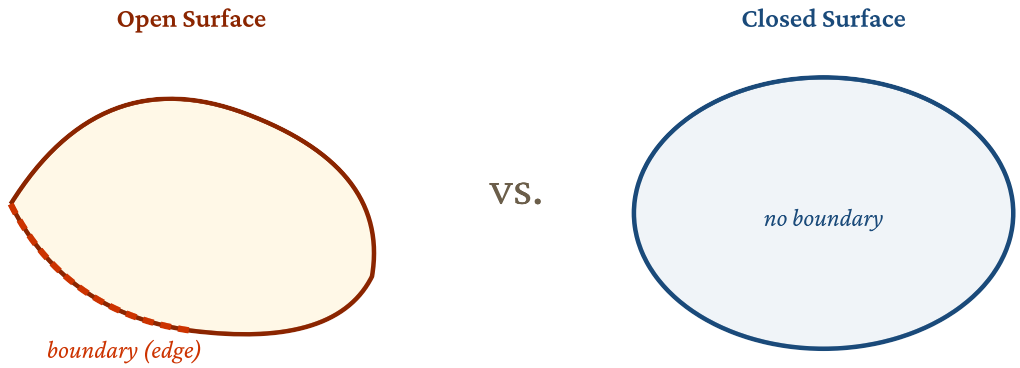
Apply **Stokes' theorem** to the left side, converting the surface integral of the curl into a line integral around the boundary ∂S:

> **STOKES' THEOREM APPLICATION**
>
> $$\oint_{\partial S} \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{A} = \mu_0 \, I_{\text{enc}}$$
>
> This is Ampère's law in integral form.

> **UNITS**
>
> Charge density: $[\rho] = \mathrm{C/m^3}$. Current density: $[\mathbf{J}] = \mathrm{A/m^2}$.

Open & Closed Manifolds

The choice of surface S in Stokes' theorem matters. A manifold (surface) is called **open** if it has a boundary edge, and **closed** if it has no boundary. A manifold can even be *clopen* — both open and closed at the same time (for example, a half-open line segment including one endpoint but not the other).



> **KEY TOPOLOGICAL FACT**
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> The boundary of a boundary is zero: $\partial(\partial M) = 0$ for any manifold M.

2. Magnetic Field of an Infinite Straight Wire

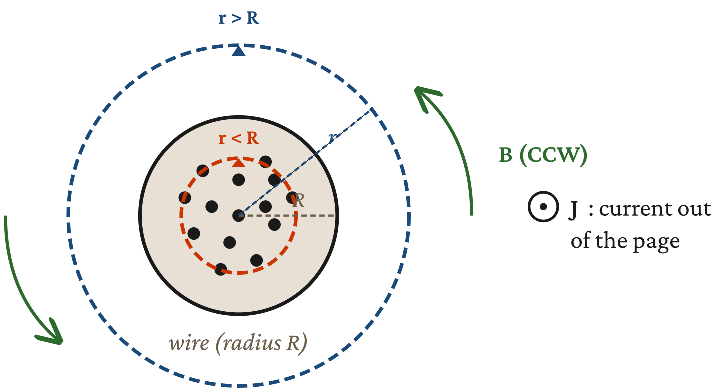
Problem: An infinite straight wire of radius R carries a total current I, distributed uniformly over its cross-sectional area. Find $\mathbf{B}$ everywhere.

Why Ampère's Law (not Biot–Savart)?

We *could* use the Biot–Savart law:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\boldsymbol{\ell}' \times \hat{\boldsymbol{r}}}{r^2}$$

But this integral would be tedious. The problem has perfect cylindrical symmetry — $\mathbf{B}$ depends only on r (the radial distance from the axis) and points in the $\hat{\boldsymbol{\phi}}$ direction. This makes Ampère's law the ideal tool.



Cross-Section of Wire with Amperian Loops

Case 1: Outside the wire ($r > R$)

Choose a circular Amperian loop of radius $r > R$ centered on the wire axis. By symmetry, $\mathbf{B}$ is tangent to the loop and constant in magnitude, so:

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = B(r) \oint d\ell = B(r) \cdot 2\pi r$$

The entire current I is enclosed, so $I_{\text{enc}} = I$. Ampère's law gives:

> **OUTSIDE THE WIRE ($r > R$)**
>
> $$B(r) \cdot 2\pi r = \mu_0 I \qquad \Longrightarrow \qquad \boxed{\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r}\,\hat{\boldsymbol{\phi}}} \quad \text{(CCW)}$$

Case 2: Inside the wire ($r < R$)

Now our Amperian loop has $r < R$, so it only encloses a fraction of the total current. Since the current is uniformly distributed, the current density is $J = I/(\pi R^2)$, and the enclosed current is:

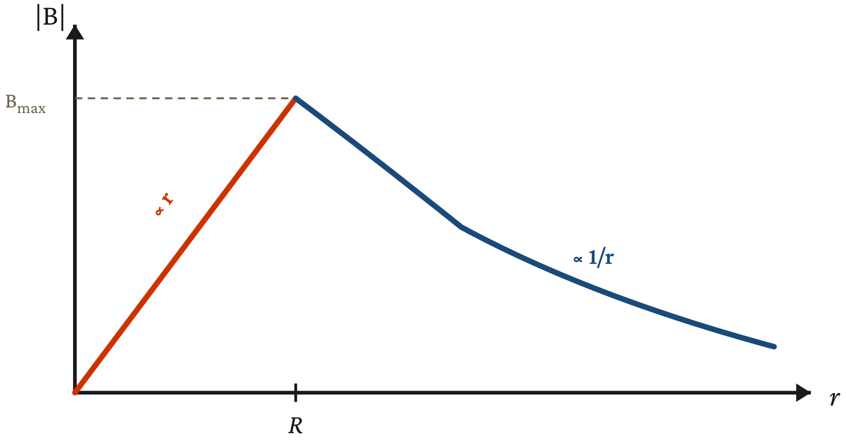
$$I_{\text{enc}} = J \cdot \pi r^2 = \frac{I}{\pi R^2} \cdot \pi r^2 = I\,\frac{r^2}{R^2}$$

The left side is the same as before, so:

INSIDE THE WIRE ($r < R$)

$$B(r) \cdot 2\pi r = \mu_0 I \,\frac{r^2}{R^2} \qquad \Longrightarrow \qquad \boxed{\mathbf{B}(r) = \frac{\mu_0 I\, r}{2\pi R^2}\,\hat{\phi}}$$

Graph of $|\mathbf{B}|$ vs. r



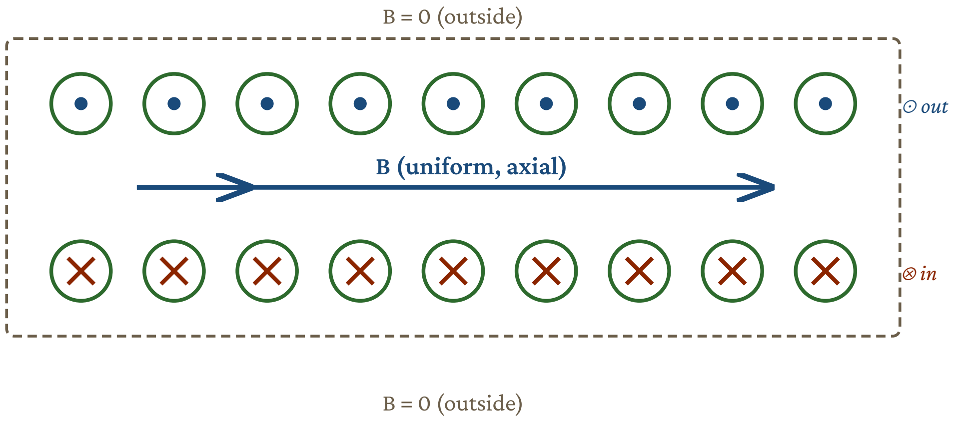
Inside the wire, $|\mathbf{B}|$ grows linearly with r. It peaks at $r = R$ with $B_{\text{max}} = \mu_0 I/(2\pi R)$, then falls off as $1/r$ outside — matching the field of an ideal line current.

3. The Infinite Solenoid

A solenoid is a coil of wire wound in a helix. An **infinite solenoid** is the idealized limit where the solenoid extends forever in both directions. Define the **loop density** (turns per unit length):

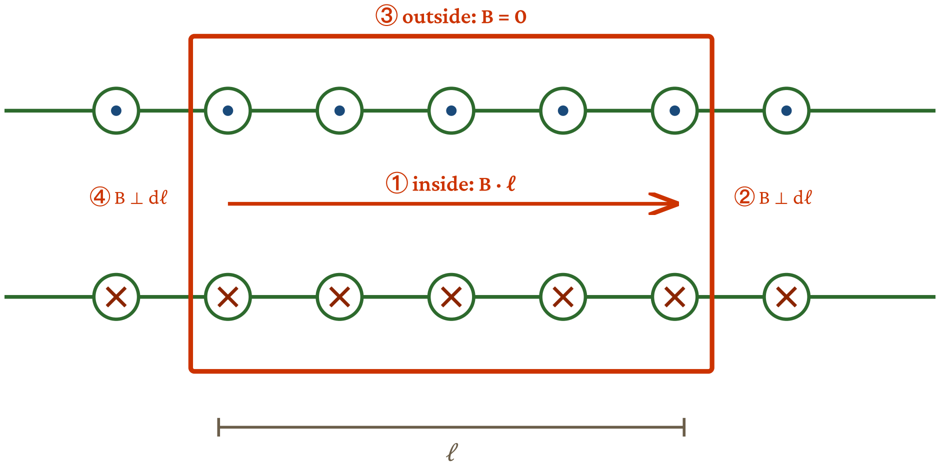
$$n = \frac{N}{L} \qquad \left[\frac{\text{turns}}{\text{m}}\right]$$

Cross-Section and Field Structure



Applying Ampère's Law with a Rectangular Loop

We draw a rectangular Amperian loop that straddles the solenoid wall — one side of length ℓ runs inside along the axis, and the opposite side runs outside.



Evaluate the loop integral side by side:

Side ① (inside, along axis): $\mathbf{B}$ is parallel to $d\boldsymbol{\ell}$, so $\int \mathbf{B} \cdot d\boldsymbol{\ell} = B\ell$.

Sides ② and ④ (perpendicular segments): $\mathbf{B}$ is perpendicular to $d\boldsymbol{\ell}$, so each contributes zero.

Side ③ (outside): The field outside an ideal infinite solenoid is zero, so this contributes zero.

The enclosed current is $I_{\text{enc}} = n\ell \cdot I$ (there are $n\ell$ turns threading through the loop). Putting it together:

$$B\ell = \mu_0 \, n\ell \, I$$

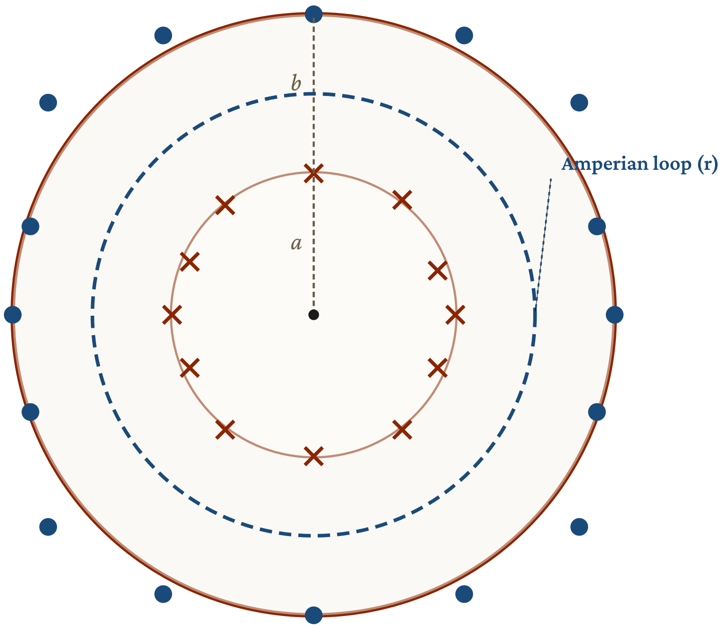
INFINITE SOLENOID

$$\boxed{\mathbf{B} = \begin{cases} \mu_0\, n\, I\, \hat{\mathbf{z}} & \text{inside} \\ 0 & \text{outside} \end{cases}}$$

The field is **uniform** inside and zero outside — it does not depend on position within the solenoid.

4. The Toroid

A toroid is a solenoid bent into a donut shape. It has inner radius a and outer radius b, with N total turns of wire carrying current I. By symmetry, $\mathbf{B}$ is everywhere tangential (in the $\hat{\boldsymbol{\phi}}$ direction around the donut) and depends only on r.



Case 1: $r < a$ (inside the hole)

An Amperian circle of radius $r < a$ encloses no current at all — every wire that passes through the loop going one way also passes through going the other way (but here we're inside the hole, so no wires thread through). Thus:

$$B \cdot 2\pi r = \mu_0 \cdot 0 \qquad \Longrightarrow \qquad B = 0$$

Case 2: $r > b$ (outside the toroid)

An Amperian circle of radius $r > b$ encloses all N turns, but the current passes through in *both* directions — each turn carries I in and I out. The net enclosed current is zero:

$$B \cdot 2\pi r = \mu_0 \cdot 0 \qquad \Longrightarrow \qquad B = 0$$

Case 3: $a < r < b$ (inside the toroid)

An Amperian circle in this region threads through all N turns, each carrying current I in the same direction through the surface:

$$B \cdot 2\pi r = \mu_0 NI$$

> **TOROID**
>
> $$\boxed{\mathbf{B}(r) = \begin{cases} 0 & r < a \\ \dfrac{\mu_0 NI}{2\pi r}\,\hat{\boldsymbol{\phi}} & a < r < b \\ 0 & r > b \end{cases}}$$

> **SOLENOID VS. TOROID**
>
> Unlike the infinite solenoid (where $\mathbf{B}$ is perfectly uniform inside), the toroid's field varies as $1/r$ within the winding region. In the limit where the toroid's radius is much larger than its cross-section ($b - a \ll (a+b)/2$), the field becomes approximately uniform and approaches $\mathbf{B} \approx \mu_0 nI$, recovering the solenoid result.