

The Toroid, Vector Potential, and Gauge Choices

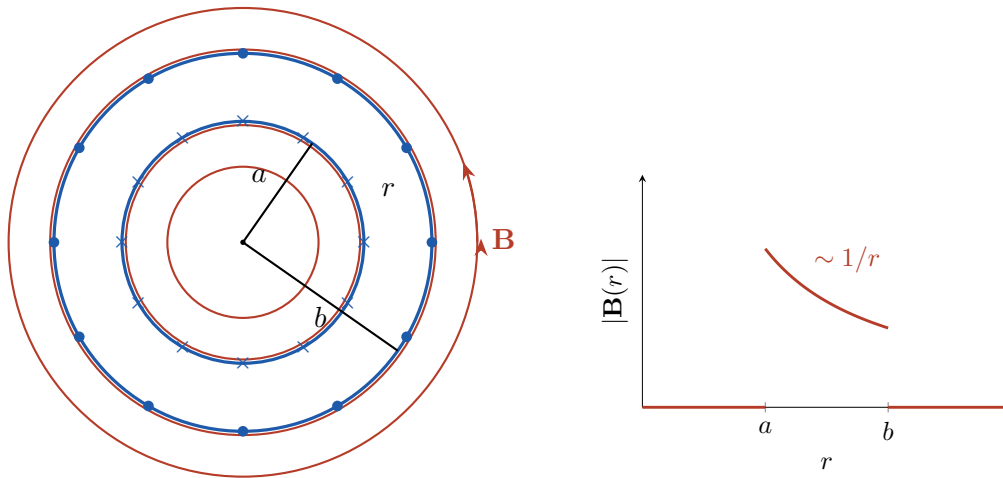
Lecture notes — April 6

1 The Toroid via Ampère's Law

Consider a toroid: a doughnut-shaped coil of N turns carrying current I , with inner radius a and outer radius b . By symmetry, \mathbf{B} circulates along $\hat{\phi}$ and depends only on the cylindrical radius r from the symmetry axis. We apply Ampère's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}},$$

on circular Amperian loops of radius r concentric with the toroid axis.



Region I: $r < a$ (inside the hole)

An Amperian loop of radius $r < a$ encloses no current ($I_{\text{enc}} = 0$). By symmetry $\mathbf{B} = B(r)\hat{\phi}$, so

$$\oint \mathbf{B} \cdot d\mathbf{l} = B(r)(2\pi r) = \mu_0 \cdot 0 \quad \Rightarrow \quad \boxed{\mathbf{B} = 0}.$$

Region III: $r > b$ (outside the toroid)

An Amperian loop of radius $r > b$ encloses each of the N turns *twice*: once going into the page on the inner edge and once coming out on the outer edge. The net enclosed current is zero, and

$$B(r)(2\pi r) = 0 \quad \Rightarrow \quad \boxed{\mathbf{B} = 0}.$$

Region II: $a < r < b$ (inside the windings)

Now the loop pierces each of the N turns once, so $I_{\text{enc}} = NI$:

$$\oint \mathbf{B} \cdot d\mathbf{l} = B(r)(2\pi r) = \mu_0 NI.$$

Solving,

$$\mathbf{B}(r) = \frac{\mu_0 N I}{2\pi r} \hat{\phi}, \quad a < r < b.$$

The direction of $\hat{\phi}$ is set by the right-hand rule applied to the winding direction.

2 The Vector Potential $\mathbf{A}(\mathbf{r})$

Gauss's law for magnetism states

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0.$$

Because the divergence of a curl vanishes identically, $\nabla \cdot (\nabla \times \mathbf{X}) \equiv 0$, we may *always* write

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}).$$

\mathbf{A} is the (magnetostatic) **vector potential**. Later we will promote it to $\mathbf{A}(\mathbf{r}, t)$.

Aside: the scalar potential

For the electric case, $\nabla \times \mathbf{E} = 0$ (in statics) implies

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}), \quad \nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}.$$

Ampère's law in terms of \mathbf{A} (no Maxwell term)

Starting from $\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$ and substituting $\mathbf{B} = \nabla \times \mathbf{A}$,

$$\mu_0 \mathbf{J}(\mathbf{r}) = \nabla \times [\nabla \times \mathbf{A}(\mathbf{r})] = \underbrace{\nabla[\nabla \cdot \mathbf{A}(\mathbf{r})]}_{\text{grad-div}} - \underbrace{\nabla^2 \mathbf{A}(\mathbf{r})}_{\text{vector Laplacian}}.$$

Gauge freedom

Recall that the scalar potential is only defined up to a constant: $V'(\mathbf{r}) = V(\mathbf{r}) + \text{const}$ gives the same \mathbf{E} , because ∇ annihilates constants. The vector potential has an even larger redundancy. Add any vector field $\lambda(\mathbf{r})$ whose curl vanishes:

$$\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \lambda(\mathbf{r}), \quad \nabla \times \lambda = 0.$$

Then

$$\mathbf{B}'(\mathbf{r}) = \nabla \times \mathbf{A}'(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) + \underbrace{\nabla \times \lambda(\mathbf{r})}_{=0} = \mathbf{B}(\mathbf{r}).$$

The simplest way to guarantee $\nabla \times \lambda = 0$ is to take $\lambda = \nabla \varphi$ for some scalar field $\varphi(\mathbf{r})$, since the curl of a gradient is identically zero. Hence the full **gauge transformation** is

$$\mathbf{A}(\mathbf{r}) \longrightarrow \mathbf{A}(\mathbf{r}) + \nabla \varphi(\mathbf{r}), \quad \mathbf{B} \text{ unchanged.}$$

3 Gauge Choices

The freedom to shift $\mathbf{A} \rightarrow \mathbf{A} + \nabla\varphi$ lets us impose one scalar condition on \mathbf{A} . A *gauge* is such a condition — a constraint on \mathbf{A} , not a physical equation.

Coulomb gauge

Choose φ so that

$$\boxed{\nabla \cdot \mathbf{A}(\mathbf{r}) = 0} \quad (\text{Coulomb gauge; a.k.a. radiation or transverse gauge}).$$

In this gauge the grad-div term in Ampère’s law drops out and we recover a Poisson equation for each Cartesian component of \mathbf{A} :

$$\nabla^2 \mathbf{A}(\mathbf{r}) = -\mu_0 \mathbf{J}(\mathbf{r}).$$

Why this works (the “tell me why” from the margin). Using the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A},$$

and Ampère’s law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ with $\mathbf{B} = \nabla \times \mathbf{A}$, we have

$$\mu_0 \mathbf{J} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

Imposing $\nabla \cdot \mathbf{A} = 0$ kills the first term, leaving $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$.

Why this is called the Coulomb gauge. Each Cartesian component satisfies the same Poisson equation that the Coulomb potential V does, $\nabla^2 V = -\rho/\epsilon_0$. By analogy with

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_{\text{all space}} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r', \quad \mathfrak{R} \equiv \mathbf{r} - \mathbf{r}',$$

we can write the magnetostatic solution immediately:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \iiint_{\text{all space}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'.$$

Other gauges

- **Lorenz gauge** (fully relativistic, 4-vector form):

$$\partial_\mu A^\mu = 0, \quad \partial_\mu A^\mu \equiv \sum_{\mu=0}^3 \partial_\mu A^\mu, \quad (\mu = 0, 1, 2, 3) \leftrightarrow (t, x, y, z).$$

This is Lorentz-covariant and is the standard choice in relativistic E&M.

- **Axial gauge:**

$$\hat{n} \cdot \mathbf{A}(\mathbf{r}) = 0,$$

where \hat{n} is a fixed spatial direction (the “axis”).

Big picture. A gauge is a *constraint* imposed on \mathbf{A} , not a law of nature. Physical fields (\mathbf{E}, \mathbf{B}) and all observables are gauge-invariant.

HW4 hint (don't use the Poisson integral)

For HW4, do *not* evaluate the volume integral above. Instead:

1. You are given \mathbf{B}_{in} and \mathbf{B}_{out} .
2. Guess $\mathbf{A}_{\text{in}}(\mathbf{r})$ and $\mathbf{A}_{\text{out}}(\mathbf{r})$ such that

$$\nabla \times \mathbf{A}_{\text{in}}(\mathbf{r}) = \mathbf{B}_{\text{in}}, \quad \nabla \times \mathbf{A}_{\text{out}}(\mathbf{r}) = \mathbf{B}_{\text{out}}.$$

3. Work in cylindrical coordinates (r, φ, z) .