Potentials in Classical and Quantum Electromagnetism

Phys 4392 — Electromagnetic Theory · Lecture notes from April 8

This guide summarizes a lecture on the role of the scalar and vector potentials ($V, \mathbf{A}$) in classical electromagnetism and quantum mechanics, the Aharonov–Bohm phase, Dirac's magnetic monopole, and how to recover $\mathbf{A}$ from a given magnetic field $\mathbf{B}$.

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1. Potentials: a computational tool or something physical?

In **classical electromagnetism**, the fields $\mathbf{E}$ and $\mathbf{B}$ are the physically meaningful quantities — they are what a device in the lab actually measures (via forces on charges). The potentials are introduced through

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A},$$

and from this point of view they look like mathematical conveniences. There is a practical tension, however:

Quantity	Easy to measure?	Easy to compute?
$\mathbf{E}$, $\mathbf{B}$	Yes	No (often hard)
V, $\mathbf{A}$	No (not directly)	Yes (often much easier)

So classically we treat V and $\mathbf{A}$ as calculational tools: solve for them first, then differentiate to get the fields.

Quantum mechanics changes this. The potentials turn out to have direct physical consequences even in regions where $\mathbf{E}$ and $\mathbf{B}$ vanish.

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2. The Aharonov–Bohm phase

A charged particle (say, an electron with charge e) traveling through a region with vector potential $\mathbf{A}$ picks up a phase:

$$\psi \;\longrightarrow\; \psi \, \exp\!\left(\frac{ie}{\hbar} \oint \mathbf{A} \cdot d\ell\right).$$

A few things to notice:

- The **probability density is unchanged**:

$$|\psi|^2 \;\longrightarrow\; |\psi|^2,$$

because the extra factor is a pure phase.

- **Absolute phase is not measurable**, but **phase differences are** — for example, in a two-slit experiment where the two paths enclose a region of nonzero $\mathbf{A}$ (even if $\mathbf{B} = 0$ along each path), the interference pattern shifts.
- This is the Aharonov–Bohm effect. It tells us $\mathbf{A}$ is not "just" a mathematical trick; gauge-invariant quantities built from it (like $\oint \mathbf{A} \cdot d\ell$, which equals the enclosed magnetic flux) carry real physics.

  Further reading. Feynman's *QED: The Strange Theory of Light and Matter* gives an accessible, pictures-and-arrows version of how phases accumulate — recommended if this is your first exposure.

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3. Dirac's magnetic monopole (sketch)

Maxwell's equations as we normally write them have $\nabla \cdot \mathbf{B} = 0$, i.e. no magnetic charges. Dirac asked: what if a magnetic monopole did exist?

Picture a monopole at the origin with field lines radiating outward like a point charge. If you try to describe this with a single smooth vector potential $\mathbf{A}$ everywhere, you run into trouble: integrating $\mathbf{A}$ around a closed loop that encircles the monopole gives a result that cannot be made to vanish as the loop shrinks from the "north" side versus the "south" side. There is a **jump discontinuity** — the famous **Dirac string** — along some line running from the monopole out to infinity.

Dirac showed that this string is unobservable *provided* electric charge is quantized, giving the celebrated Dirac quantization condition

$$\frac{e\,g}{\hbar c} = \frac{n}{2}, \qquad n \in \mathbb{Z},$$

where g is the magnetic charge. A single monopole anywhere in the universe would explain why all electric charges are integer multiples of e.

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4. Recovering A from B

Problem. Given $\mathbf{B}(\mathbf{r})$ everywhere, find a vector potential $\mathbf{A}(\mathbf{r})$ such that $\mathbf{B} = \nabla \times \mathbf{A}$.

Starting point. In magnetostatics, Ampère's law reads

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

and the standard integral expression for $\mathbf{A}$ in Coulomb gauge is

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi} \iiint \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, d^3 r'.$$

Here $\mathbf{r}$ is the **field point** (where we want $\mathbf{A}$) and $\mathbf{r}'$ is the **dummy variable of integration** ranging over all space.

Substitute Ampère's law. Using $\mathbf{J} = \dfrac{1}{\mu_0}\,\nabla' \times \mathbf{B}$ (where ∇' denotes derivatives with respect to the source coordinate $\mathbf{r}'$):

$$\mathbf{A}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\iiint \frac{1}{|\mathbf{r}-\mathbf{r}'|}\cdot\frac{1}{\mu_0}\,\nabla'\times\mathbf{B}(\mathbf{r}')\,d^3r'.$$

The μ_0's cancel, leaving

$$\boxed{\mathbf{A}(\mathbf{r}) \;=\; \frac{1}{4\pi}\iiint \frac{\nabla'\times\mathbf{B}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r'.}$$

Why this works

Two ingredients from magnetostatics make the derivation legitimate:

1. **Ampère's law** $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ lets us trade the source current $\mathbf{J}$ for the curl of a field we already know.
2. The **Green's function for the Laplacian**,

$$\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\,\delta^{(3)}(\mathbf{r}-\mathbf{r}'),$$

is what makes $\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{J}(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|\,d^3r'$ satisfy $\nabla^2\mathbf{A} = -\mu_0\mathbf{J}$ in Coulomb gauge ($\nabla\cdot\mathbf{A} = 0$).

Combining these: the potential written in terms of $\nabla'\times\mathbf{B}$ is just the Coulomb-gauge potential with Ampère's law substituted in. It assumes magnetostatic conditions ($\partial_t \to 0$) and that $\mathbf{B}$ falls off fast enough at infinity for the integral to converge and for boundary terms from integration by parts to vanish.

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5. Quick-reference summary

- Classically, $(V, \mathbf{A})$ are computational tools; only $(\mathbf{E}, \mathbf{B})$ are directly measurable.
- Quantum mechanically, $\mathbf{A}$ shifts the phase of a charged particle's wavefunction by $(e/\hbar)\oint \mathbf{A}\cdot d\ell$. Probability density is unchanged, but interference patterns can shift (Aharonov–Bohm).
- Hypothetical magnetic monopoles force a Dirac string in $\mathbf{A}$; demanding it be unobservable quantizes electric charge.
- Given $\mathbf{B}$ everywhere in magnetostatics,

$$\mathbf{A}(\mathbf{r}) = \frac{1}{4\pi}\iiint \frac{\nabla'\times\mathbf{B}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3r',$$

  which follows from the standard Coulomb-gauge formula plus Ampère's law.

Compiled from lecture notes, April 8. Corrections welcome.