Potentials, Faraday's Law, and the Multipole Expansion of $\mathbf{A}(\mathbf{r})$

Phys 4392 — Electromagnetic Theory · Lecture notes from April 13

This guide covers why the scalar potential alone is not enough to describe time-varying electric fields, the logical chain from sources to potentials to fields, the source-point/field-point geometry behind the vector potential, and the multipole expansion of $\mathbf{A}(\mathbf{r})$ for a localized current loop. It closes with the pattern of allowed multipoles for electric and magnetic sources.

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1. Why you cannot write $\mathbf{E} = -\nabla V$ alone

Throughout this section, all scalar and vector fields are taken to be functions of $(\mathbf{r}, t)$ — we are **not** in the static limit.

Last lecture gave us Faraday's law,

$$\nabla \times \mathbf{E} \;=\; -\frac{\partial \mathbf{B}}{\partial t},$$

which says that a changing magnetic field produces an EMF around any closed loop. The classic picture: push a bar magnet through a coil and the changing flux drives a current.

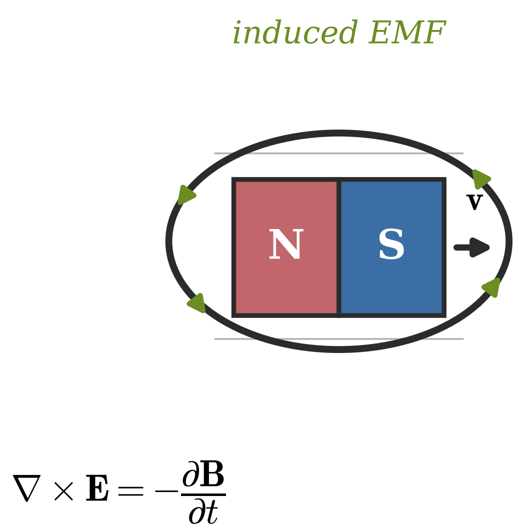

Figure 1: Bar magnet being pushed through a loop — a changing $\mathbf{B}$ produces an EMF around the loop, i.e. Faraday's law.

Now suppose we tried to import the electrostatic relation $\mathbf{E} = -\nabla V$ into this time-dependent setting. Taking the curl of both sides,

$$\nabla \times \mathbf{E} \;=\; -\nabla \times (\nabla V) \;=\; 0,$$

because the curl of any gradient vanishes identically. **That contradicts Faraday's law** — with only a scalar potential, we have thrown away EMFs, induction, transformers, and most of 19th-century electrical engineering.

The fix is to promote the vector potential $\mathbf{A}$ to a full partner of V. The correct definitions are

$$\boxed{\mathbf{E} \;=\; -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} \;=\; \nabla \times \mathbf{A}.}$$

Taking the curl of the first equation,

$$\nabla \times \mathbf{E} \;=\; -\nabla \times (\nabla V) \;-\; \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \;=\; 0 \;-\; \frac{\partial \mathbf{B}}{\partial t} \;=\; -\frac{\partial \mathbf{B}}{\partial t},$$

and Faraday's law is recovered automatically. Geometrically, V still controls the "static" piece of $\mathbf{E}$ — think of the familiar high-V to low-V arrow pointing from a 1000 V equipotential to an 80 V one — but the $-\partial \mathbf{A}/\partial t$ piece carries the induction effects that a pure gradient could never reproduce.

The source → potential → field chain

Putting this together, the logical structure of both electrostatics and magnetostatics (and, with retarded potentials, electrodynamics as well) is a three-cornered relationship:

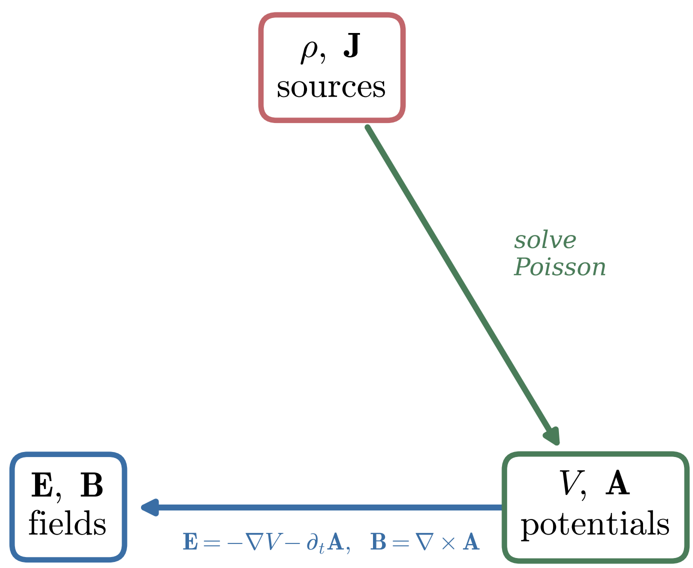

Figure 2: The sources–potentials–fields triangle. We compute downhill: given $\rho, \mathbf{J}$, solve for $V, \mathbf{A}$; then differentiate to get $\mathbf{E}, \mathbf{B}$.

- **Top:** the sources $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$.
- **Right:** the potentials $V(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$, obtained from the sources by solving Poisson-type equations.

- **Left:** the physical fields $\mathbf{E}, \mathbf{B}$, obtained from the potentials by differentiation.

We solve for the potentials first (because their equations are easier) and only then take derivatives to recover the fields.

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2. Source point, field point, and the geometry of $\mathbf{A}(\mathbf{r})$

The fields are functions of $\mathbf{r}$, **not** $\mathbf{r}'$. (If the sources also depend on time, the story becomes radiation and we need retarded potentials — that is for a later lecture.) Before writing down the multipole expansion, be very careful with notation — mixing up $\mathbf{r}$ and $\mathbf{r}'$ is the single most common source of sign errors in this subject.

- $\mathbf{r}'$ is the **source point** — the dummy variable of integration, ranging over wherever the current lives.
- $\mathbf{r}$ is the **field point** — where we want to evaluate $\mathbf{A}$.
- $\mathbf{r} - \mathbf{r}'$ is the **separation vector** from source to field point, with magnitude $|\mathbf{r} - \mathbf{r}'|$.

The picture to keep in your head is a localized current loop $\mathcal{C}$ carrying steady current I, with origin O inside the loop, $\mathbf{r}'$ sweeping over the wire, and $\mathbf{r}$ pointing from the origin to some distant observation point:

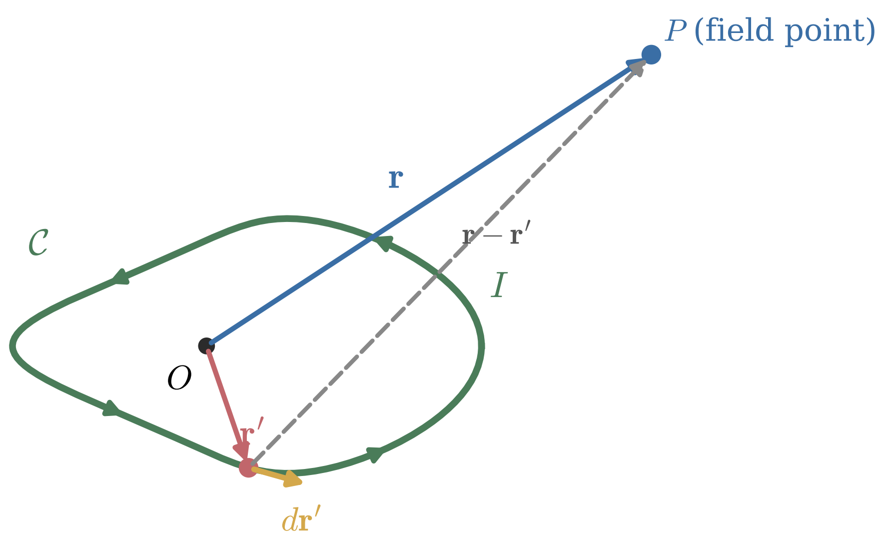

Figure 3: Source point $\mathbf{r}'$ sweeps around the current loop $\mathcal{C}$; the field point $\mathbf{r}$ lives outside the loop. The oriented line element is $d\mathbf{r}' = d\ell$, and the separation vector $\mathbf{r} - \mathbf{r}'$ is what appears in the integrand of $\mathbf{A}(\mathbf{r})$.

The Coulomb-gauge vector potential for a general current density is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\iiint_{\text{all space}} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, d^3r'.$$

For current confined to a thin wire loop of current I, the volume integral collapses to a line integral (replace $\mathbf{J}(\mathbf{r}')\, d^3r'$ by $I\, d\mathbf{r}'$):

$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_{\mathcal{C}} \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}.}$$

This is the expression we will expand in the next section.

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3. Expanding $1/|\mathbf{r}-\mathbf{r}'|$ for $r' < r$

Suppose the observer is far from the loop, so $r > r'$. Let θ' be the angle between $\mathbf{r}$ and $\mathbf{r}'$. Then

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')} = \sqrt{r^2 + (r')^2 - 2rr'\cos\theta'}.$$

Factor out r and define $\varepsilon \equiv r'/r < 1$:

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\frac{1}{\sqrt{1-2\varepsilon\cos\theta' + \varepsilon^2}}.$$

The denominator looks unfamiliar, but it is actually the **generating function for the Legendre polynomials** — expanding by the binomial theorem (or just looking up the identity) gives

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} P_\ell(\cos\theta').$$

The convergence condition $\varepsilon < 1$, i.e. $r' < r$, is essential — it is what makes this a **far-field** expansion. The first few Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \tfrac{1}{2}(3x^2-1), \quad P_3(x) = \tfrac{1}{2}(5x^3-3x), \ \ldots$$

Plugging this expansion into the boxed formula for $\mathbf{A}(\mathbf{r})$ and pulling the r-dependent pieces outside the integral gives the multipole expansion written out explicitly term by term:

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0 I}{4\pi}\left[\underbrace{\frac{1}{r}\oint_{\mathcal{C}} d\mathbf{r}'}_{\ell=0\ \text{(vanishes)}} + \underbrace{\frac{1}{r^2}\oint_{\mathcal{C}} r'\cos\theta'\, d\mathbf{r}'}_{\ell=1\ \text{magnetic dipole}} + \underbrace{\frac{1}{r^3}\oint_{\mathcal{C}} (r')^2\left(\tfrac{3}{2}\cos^2\theta' - \tfrac{1}{2}\right) d\mathbf{r}'}_{\ell=2\ \text{magnetic quadrupole}} + \cdots\right].$$

Each term falls off one power of r faster than the previous, so far from the loop the lowest surviving ℓ dominates.

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4. Why the magnetic monopole term vanishes

Look at the $\ell = 0$ piece. It is

$$\frac{\mu_0 I}{4\pi r} \oint_{\mathcal{C}} d\mathbf{r}',$$

and the integral is just the **net vector displacement around a closed loop** — which is zero. So

$$\oint_{\mathcal{C}} d\mathbf{r}' \;=\; \mathbf{0} \qquad \Longrightarrow \qquad \mathbf{A}^{(\ell=0)} \;=\; \mathbf{0}.$$

This is one very concrete way of saying "there are no magnetic monopoles": the would-be magnetic monopole contribution to $\mathbf{A}$ from any closed current loop is identically zero, because you cannot end up somewhere other than where you started after tracing out the loop. The same conclusion follows more generally for any steady, localized $\mathbf{J}$ (via $\int \mathbf{J}\, d^3r' = 0$, which is a consequence of $\nabla \cdot \mathbf{J} = 0$ plus integration by parts).

The leading magnetic term is therefore the dipole ($\ell = 1$), not the monopole. For a planar loop this gives the familiar

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) \;=\; \frac{\mu_0}{4\pi}\,\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}, \qquad \mathbf{m} \;=\; I\,\mathbf{A}_{\text{loop}},$$

where $\mathbf{A}_{\text{loop}}$ is the (vector) area of the loop. After that comes the magnetic quadrupole ($\ell = 2$), and so on.

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5. Electric vs. magnetic multipoles side by side

The same $|\mathbf{r} - \mathbf{r}'|$ expansion, applied to the scalar potential $V(\mathbf{r})$ of a charge distribution $\rho(\mathbf{r}')$, gives the electric multipole expansion with moments

$$Q = \iiint \rho\, d^3r', \qquad \mathbf{p} = \iiint \mathbf{r}'\rho\, d^3r', \qquad Q_{ij} = \iiint (3r_i' r_j' - r'^2\delta_{ij})\rho\, d^3r', \; \ldots$$

For **electric** sources all three moments exist. For **magnetic** sources the monopole is absent and the dipole $\mathbf{m} = I\mathbf{A}_{\text{loop}}$ leads. The rogues' gallery of charge configurations and their field patterns looks like this:

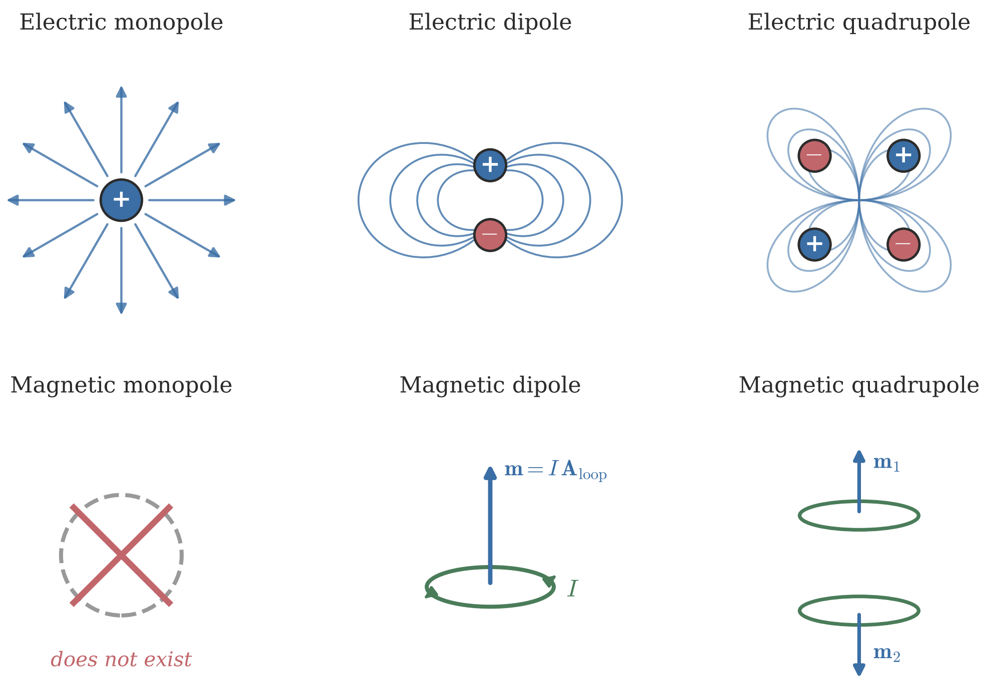

Figure 4: Pole representations. Top row — electric sources: monopole (single charge, radial field), dipole (two opposite charges with the familiar two-lobed field), quadrupole (four charges in a square pattern with a four-lobed field). Bottom row — magnetic sources: no monopole, the dipole is a current loop with $\mathbf{m} = I\,\mathbf{A}_{\text{loop}}$, and the quadrupole is a pair of antiparallel current loops.

Summary table

Source type	Monopole ($\ell = 0$)	Dipole ($\ell = 1$)	Quadrupole ($\ell = 2$)
Electric	Q (point charge)	$\mathbf{p} = q\mathbf{d}$	Q_{ij}
Magnetic	— (does not exist)	$\mathbf{m} = I\mathbf{A}_{\text{loop}}$	$\mathbf{m}_{ij}$

A useful mental picture

- **Electric monopole:** a single point charge q. Potential $\propto 1/r$, field $\propto 1/r^2$.
- **Electric dipole:** $+q$ and $-q$ separated by a small vector $\mathbf{d}$, giving $\mathbf{p} = q\mathbf{d}$. The "ideal" dipole limit is $q \to \infty$, $d \to 0$ with $qd = p$ held fixed.
- **Electric quadrupole:** $+q, -q, -q, +q$ on a square — total charge zero, total dipole zero, leading surviving term is $\ell = 2$.

Each higher multipole can be built by taking the previous configuration, making a copy, flipping its sign, and displacing it by a small vector.

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6. Quick-reference summary

- $\mathbf{E} = -\nabla V$ alone kills Faraday's law. The correct time-dependent definition is $\mathbf{E} = -\nabla V - \partial \mathbf{A}/\partial t$ with $\mathbf{B} = \nabla \times \mathbf{A}$, and this automatically reproduces $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$.
- The logical chain is sources $\rho, \mathbf{J} \to$ potentials $V, \mathbf{A} \to$ fields $\mathbf{E}, \mathbf{B}$.
- Source point $\mathbf{r}'$, field point $\mathbf{r}$; for a current loop, $\mathbf{A}(\mathbf{r}) = \dfrac{\mu_0 I}{4\pi} \oint_{\mathcal{C}} d\mathbf{r}'/|\mathbf{r} - \mathbf{r}'|$.
- For $r' < r$, $\dfrac{1}{|\mathbf{r} - \mathbf{r}'|} = \dfrac{1}{r} \sum_{\ell=0}^{\infty} \left(\dfrac{r'}{r}\right)^{\ell} P_\ell(\cos\theta')$.
- The $\ell = 0$ magnetic term vanishes because $\oint_{\mathcal{C}} d\mathbf{r}' = \mathbf{0}$ — no magnetic monopole.
- The leading magnetic term is the dipole, $\mathbf{m} = I\mathbf{A}_{\text{loop}}$; electric sources have monopole, dipole, quadrupole, ... all generically nonzero.

Compiled from lecture notes, April 13. Corrections welcome.