# Magnetization, the Auxiliary Field *H*, and the Magnetized Cylinder

PHYS 4392 — Electromagnetism | Lecture of April 20, 2026

## 1. Recap: where free and bound currents enter

Last lecture we split the total current into a free part (whatever we drive through wires and circuits) and a bound part (whatever the medium contributes via its microscopic magnetic moments). Ampère's law in matter then reads

$$\nabla \times \vec{B} = \mu_0 \vec{J}_{\text{total}} = \mu_0 \left[\vec{J}_{\text{free}} + \vec{J}_{\text{bound}}\right] = \mu_0 \nabla \times \vec{H} + \mu_0 \nabla \times \vec{M},$$

with the by-now familiar identifications

$$\nabla \times \vec{H} = \vec{J}_{\text{free}}, \qquad \nabla \times \vec{M} = \vec{J}_{\text{bound}}.$$

Outside the material both $M$ and $J_{\text{bound}}$ vanish, so $B = \mu_0 H$ there. Inside, the two pieces are genuinely different and must be treated separately.

## 2. Bulk and Surface Bound Currents

The bound current splits naturally into a volume part and a surface part. In the bulk, the magnetization gives a current density

$$\vec{J}_{\text{bound}} = \nabla \times \vec{M},$$

while at the boundary of the material a surface current density appears,

$$\vec{K}_{\text{bound}} = \vec{M} \times \hat{n},$$

where $\hat{n}$ is the outward unit normal at the surface. Physically, the microscopic Ampèrian loops cancel one another in the bulk except at the boundary, where the unpaired pieces sum to a net circulating current.

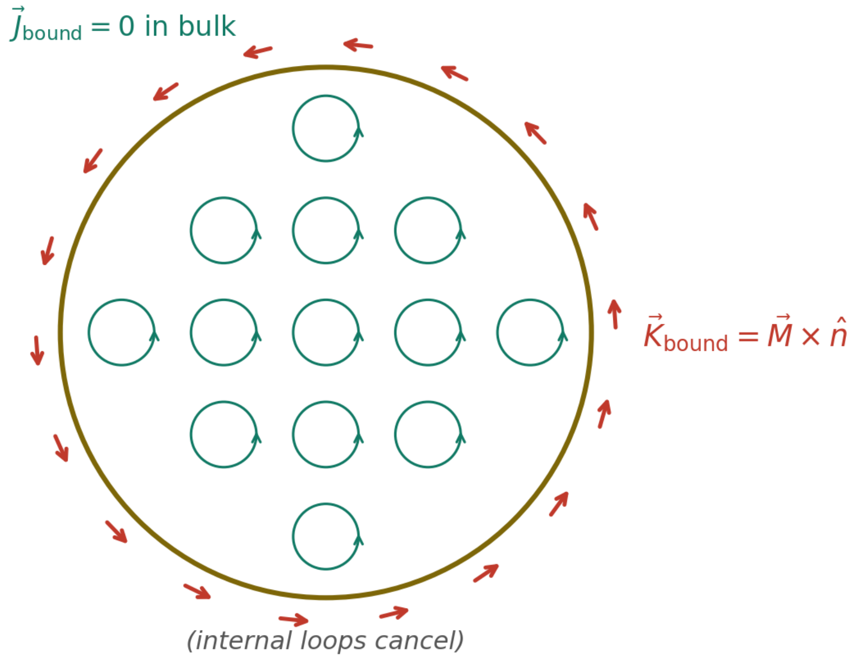


*Figure 1. End-on view of a uniformly magnetized cylinder. The microscopic current loops cancel pairwise in the interior, leaving a net surface current $K_{bound}$ circulating around the rim.*

## 3. Worked Example: Infinitely Long Magnetized Cylinder

Consider an infinite cylinder of radius $a$, uniformly magnetized along its axis:

$$\vec{M}(\vec{r}) \;=\; M_0\,\Theta(a - s)\,\hat{z}\,,$$

where $s$ is the cylindrical radial coordinate and $\Theta$ is the unit step. The cylinder has three faces: the top cap ($\hat{n} = \hat{z}$), the bottom cap ($\hat{n} = -\hat{z}$), and the lateral surface ($\hat{n} = \hat{s}$).

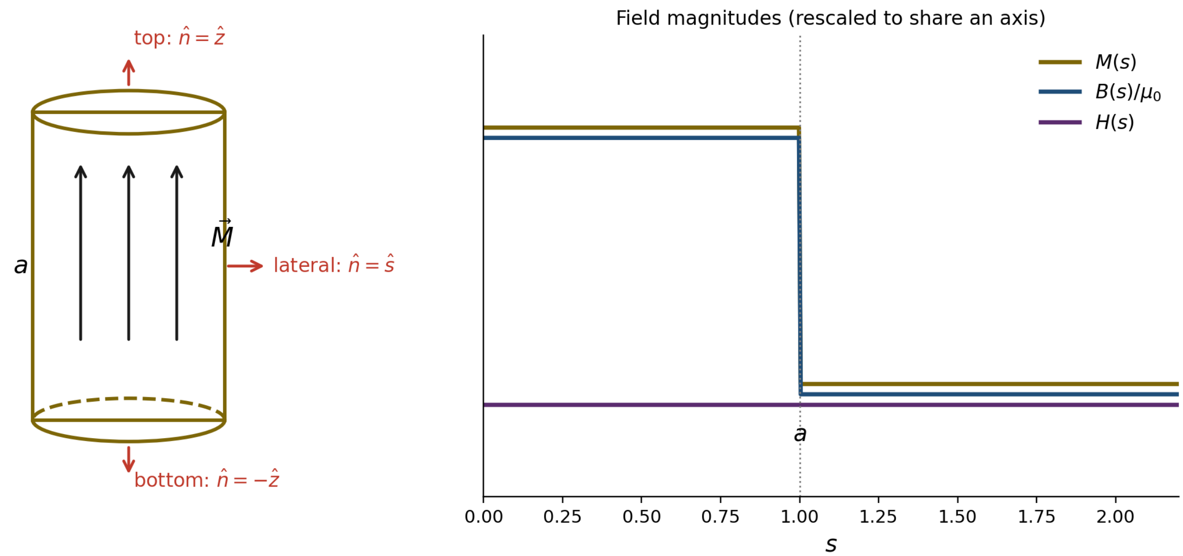


*Figure 2. Left: the uniformly magnetized cylinder with the three outward normals indicated. Right: the magnetization M(s), the resulting field B(s), and the auxiliary field H(s). The vertical axes are not to a common scale — they have different units — but are arranged to show the qualitative shapes.*

## Bulk bound current

Since $M$ is uniform inside the cylinder, its curl vanishes there:

$$\vec{J}_{\text{bound}} = \nabla \times \vec{M} = 0 \quad (s < a).$$

## Surface bound current

On the lateral surface, $\hat{n} = \hat{s}$ and

$$\vec{K}_{\text{bound}} = \vec{M} \times \hat{n} = M_0 \hat{z} \times \hat{s} = M_0 \hat{\varphi} \qquad (s = a \text{ understood}).$$

On the top cap $M$ is parallel to $\hat{n} = \hat{z}$, and on the bottom cap antiparallel to $\hat{n} = -\hat{z}$; in both cases $M \times \hat{n} = 0$. So all the bound current sits on the lateral surface, circulating in $\hat{\varphi}$.

> *If we want to bury this surface current inside a single bulk expression, we can write it with a delta function as $\mathbf{J}_{bound}(\mathbf{r}) = M_0\, \delta(s - a)\, \hat{\varphi}$. The change in M happens only at s = a, and the delta enforces that.*

## Solving for B and H by analogy with the solenoid

The bound surface current is exactly that of an infinite solenoid with $nI \to M_0$. Inside an ideal infinite solenoid the field is uniform and axial; outside, it vanishes. Therefore, inside ($s < a$):

$$\vec{B} = \mu_0 M_0 \hat{z},$$

and outside ($s > a$):

$$\vec{B} = 0 .$$

There are no free currents anywhere, so $\nabla \times H = 0$ in this geometry, and one can show $H = 0$ throughout. Equivalently,

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M} = 0,$$

since inside, $B/\mu_0 = M$, and outside both pieces are zero individually.

### Dimensional sanity check

It is worth checking units explicitly the first time one writes a delta-function current density:

| Quantity | SI units |
|---|---|
| $\mu_0$ | H/m = T·m/A |
| $J$ (volume current density) | $A/m^2$ |
| $K$ (surface current density) | A/m |
| $I$ | A |
| $\delta(s-a)$ | 1/m |
| $n$ (turns per unit length) | 1/m |

These confirm that $K_{bound} = M_0\,\hat{\varphi}$ has units of A/m and that $J_{bound} = M_0\,\delta(s-a)\,\hat{\varphi}$ has units of $A/m^2$, as required.

## 4. A Subtlety: *H* Need Not Vanish When $J_{free} = 0$

It is tempting, on seeing $\nabla \times H = J_{free}$, to conclude that $H = 0$ wherever there is no free current. That is wrong. Vanishing curl does not imply a vanishing field; it only constrains the rotational part. $H$ can still have a divergence, and it generically does in finite magnetized objects.

Consider a finite uniformly magnetized bar (a permanent magnet) with no free current. Inside the bar:

$$\nabla \times \vec{H} = \vec{J}_{free} = 0, \qquad \nabla \cdot \vec{H} \neq 0, \qquad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}.$$

The infinite cylinder of Section 3 has $H = 0$ because the geometry is translation-invariant and the integral curves of $H$ close at infinity. A finite bar magnet does not have that luxury, and one finds $H$ pointing from the north end to the south end inside the magnet.

## 5. Forces and Torques on a Point Magnetic Dipole

Once we know the field of a dipole, the natural next question is how an external field acts on it. For a point magnetic moment *m* in an external field *B*(*r*):

$$\vec{\tau} = \vec{m} \times \vec{B} ,$$

the torque vanishes when *m* is parallel to *B*, which is the stable alignment. The force requires a gradient,

$$\vec{F} = (\vec{m} \cdot \nabla) \vec{B} = \left( m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z} \right) \vec{B} ,$$

so a uniform *B* produces zero net force — only torque. A dipole is pulled toward regions where the field is stronger only when there is a spatial variation.

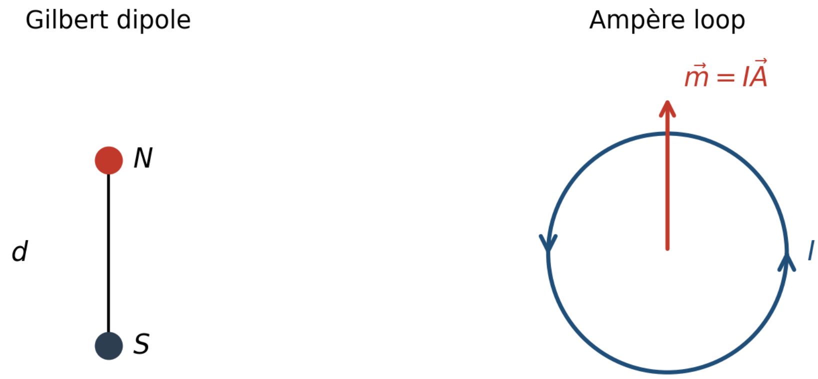


*Figure 3. A point magnetic dipole in a uniform external field. The torque m×B tends to align m with B; in a uniform field there is no net translational force.*

### Floating-needle compass

A pleasant low-tech demonstration: magnetize a sewing needle and rest it on a small piece of cork floating on water. Free of mechanical pivot friction, the needle aligns with the local horizontal component of the Earth's magnetic field — a literal point-dipole-in-an-external-field experiment, slowed to laboratory time by water viscosity.

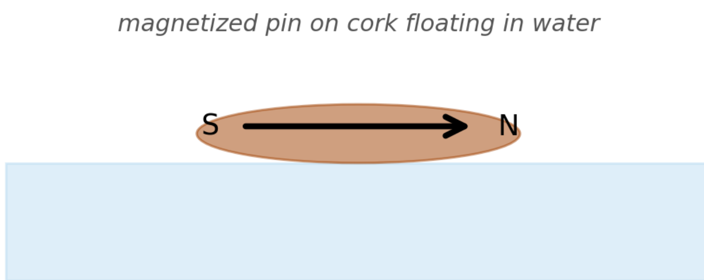


*Figure 4. Magnetized needle on a cork raft; the dipole moment m aligns with the horizontal component of the Earth's field.*