Test 2 Makeup: Multipole Expansion, Displacement Current, and Coaxial Cables

PHYS 4392 — Electromagnetism | Lecture of April 24, 2026

This session works through three problems modeled after the most common Test 2 themes: building a multipole expansion for the potential of a displaced charge, applying the Ampère–Maxwell law to a charging capacitor, and finding the magnetic field and stored energy of a coaxial cable. The unifying skill is recognizing which symmetry to exploit and which integral form (potential expansion, Stokes' theorem, or Ampère's law) is most efficient.

1. Q1: Multipole Expansion of the Electrostatic Potential

Reference case: charge at the origin

For a single point charge q at the origin, the potential at the field point **r** is the familiar Coulomb form,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}|} \equiv \frac{Kq}{|\vec{r}|},$$

where $K = 1/(4\pi\varepsilon_0)$ collects the prefactor for the rest of this section.

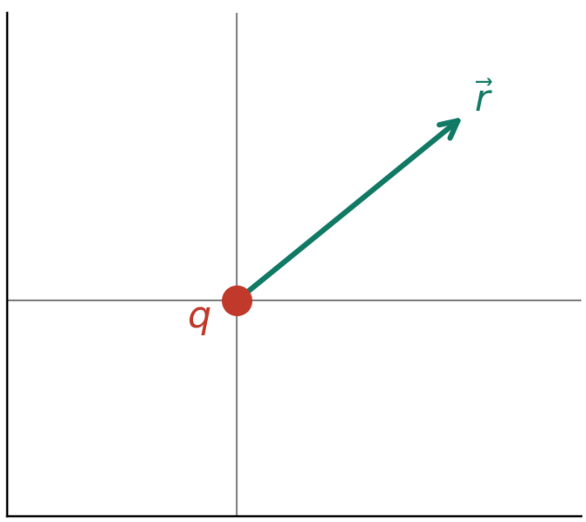

*Figure 1. A point charge at the origin and a generic field point at **r**.*

Charge displaced from the origin

Now move the charge to position **a**. The exact potential is

$$V(\vec{r}) = \frac{Kq}{|\vec{r} - \vec{a}|},$$

and far from the source (|**r**| >> |**a**|) we expand in powers of |**a**|/|**r**|. The leading two terms are

$$V(\vec{r}) \approx \frac{Kq}{|\vec{r}|} + \frac{K\vec{p}\cdot\hat{r}}{|\vec{r}|^2} + \mathcal{O}\left(\frac{1}{|\vec{r}|^3}\right),$$

where **p** = q**a** is the dipole moment of the configuration. The first term is the *monopole* and decays as $1/r$; the second is the *dipole* and decays as $1/r^2$; higher multipoles (quadrupole, octupole, ...) decay as $1/r^3$ and faster.

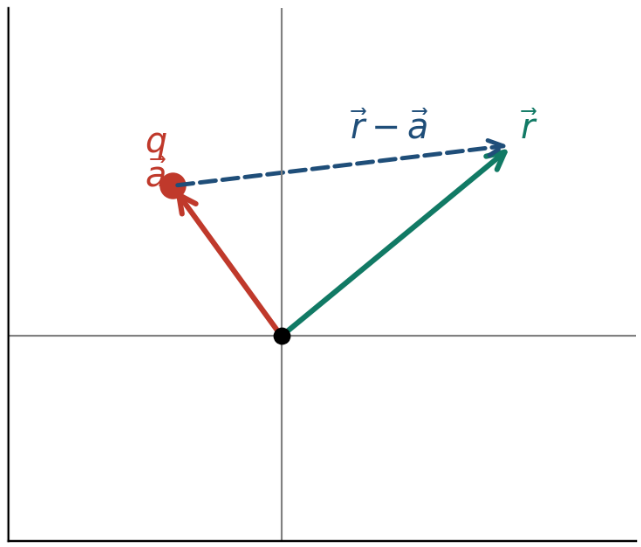

*Figure 2. Charge displaced to **a**; the relevant separation is **r**−**a**. For |**r**| >> |**a**| the potential is approximated by a few leading multipole terms.*

Two-charge configurations

For a system of charges q_i at positions $\mathbf{a}_i$, the potential is the superposition. The monopole moment is Σq_i and the dipole moment is $\mathbf{p} = \Sigma q_i \mathbf{a}_i$. Two simple two-charge cases are illuminating.

Case A: physical dipole — +q at +a, −q at −a

The total charge vanishes, so the monopole term is zero. The dipole moment is

$$\vec{p} = (+q)(+\vec{a}) + (-q)(-\vec{a}) = 2q\,\vec{a},$$

and the leading large-distance behavior of the potential is purely dipolar:

$$V(\vec{r}) \approx \frac{2Kq\,\vec{a}\cdot\hat{r}}{|\vec{r}|^2}.$$

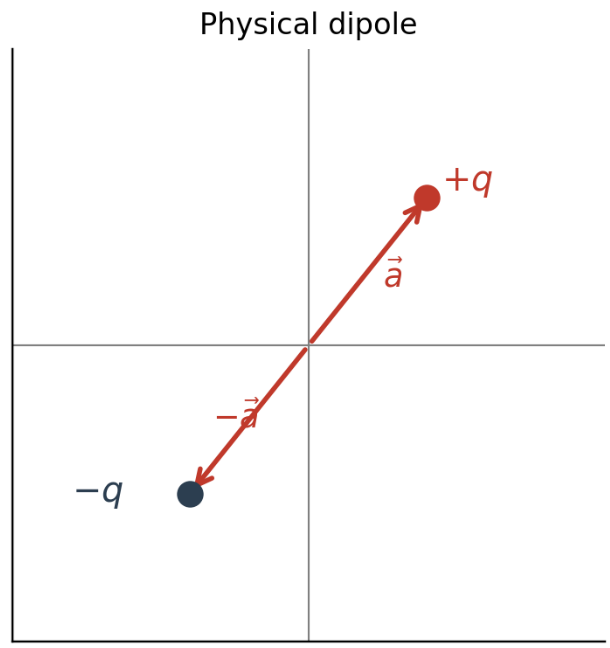

Figure 3. The physical dipole. Net charge is zero, so the monopole contribution vanishes; the leading behavior is the dipole term decaying as $1/r^2$.

Case B: two like charges — +q at +a, +q at −a

Now the dipole moment vanishes by cancellation,

$$\vec{p} \;=\; (+q)(+\vec{a}) + (+q)(-\vec{a}) \;=\; 0\,,$$

but the total charge is $2q$, so the leading term is the monopole:

$$V(\vec{r}) \;\approx\; \frac{2Kq}{|\vec{r}|}\,.$$

Two like charges (no dipole moment)

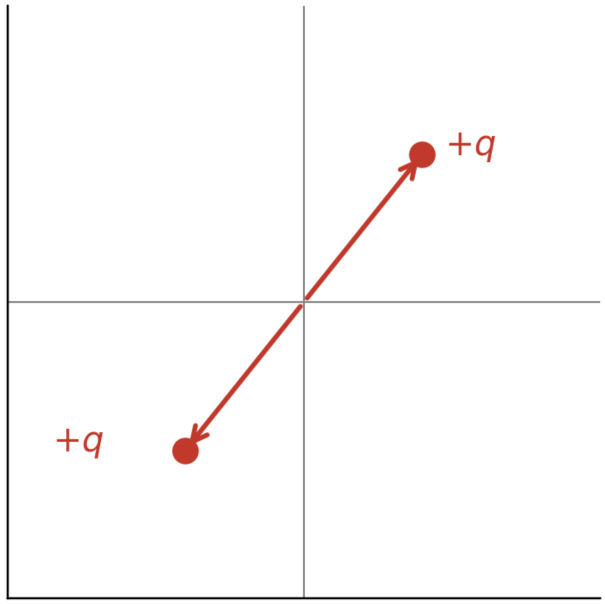

Figure 4. Two like charges. The dipole moment is zero, so the leading behavior is the monopole term decaying as 1/r.

Rule of thumb: the leading nonvanishing multipole sets the asymptotic decay. If the total charge is nonzero, the potential goes like $1/r$. If the total charge is zero but the dipole moment is not, it goes like $1/r^2$. If both vanish, look to the quadrupole: $1/r^3$, and so on.

General two-charge configuration

For arbitrary charges q_a and q_b at positions **a** and **b**, the multipole moments are

$$Q = q_a + q_b\,, \qquad \vec{p} = q_a\vec{a} + q_b\vec{b}\,,$$

and one continues with the quadrupole tensor and higher moments if needed.

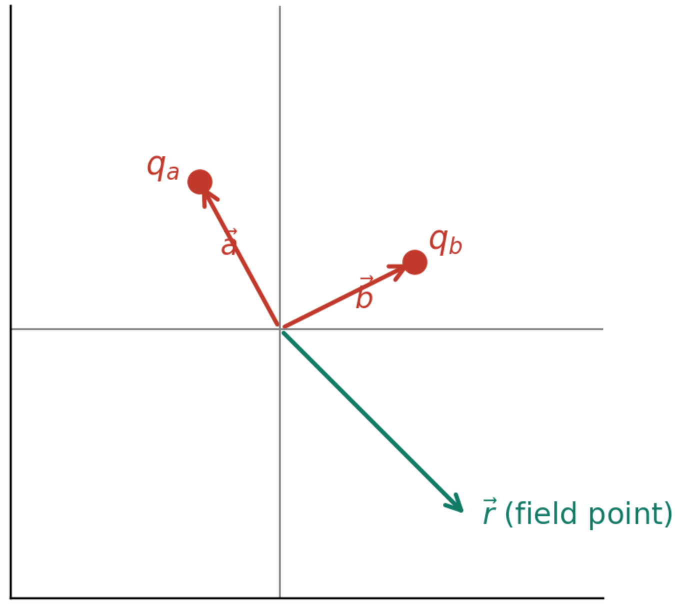

*Figure 5. General two-charge configuration. The field point is **r**; the source positions are **a** and **b**.*

2. Q2: Charging Capacitor and the Ampère–Maxwell Law

Consider an RC circuit driven by an EMF ε, with a parallel-plate capacitor of plate radius a and a resistor R. At $t = 0$ the switch closes and the capacitor begins to charge. We want the magnetic field both inside (between the plates) and outside (around the wire), and this is where the displacement current earns its keep.

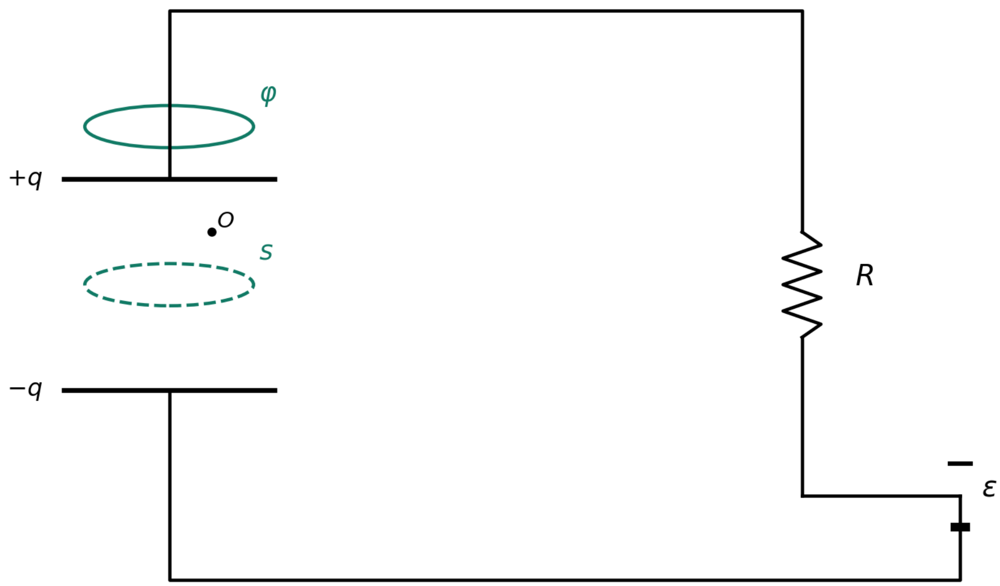

Figure 6. RC circuit with parallel-plate capacitor. Two Amperian loops are sketched: a solid one in the wire region (around the top plate's lead) and a dashed one between the plates.

Charging current and accumulated charge

Kirchhoff's loop law gives the standard RC charging response. The current through the wire is

$$I(t) \;=\; \frac{\varepsilon}{R}\, e^{-t/RC}\,,$$

and integrating from 0 to t gives the charge on the upper plate,

$$q(t) \;=\; \int_0^t I(t')\,dt' \;=\; C\,\varepsilon\left(1 - e^{-t/RC}\right).$$

Both $I(t)$ and $q(t)$ appear in what follows; $I(t)$ sets the field outside the capacitor, while $q(t)$ sets the time-varying electric field between the plates and hence the displacement current.

Ampère–Maxwell law

The integral form of the Ampère–Maxwell law is

$$\oint_{\partial S} \vec{B}\cdot d\vec{\ell} \;=\; \mu_0\, I_{\text{enc}} \;+\; \mu_0 \varepsilon_0 \frac{d}{dt}\int_S \vec{E}\cdot d\vec{S}\,.$$

The surface S is any open surface bounded by the loop. Maxwell's key insight: the second term — the displacement current — is what makes the law consistent for surfaces that pass between capacitor plates (where no real current flows, but a changing E does).

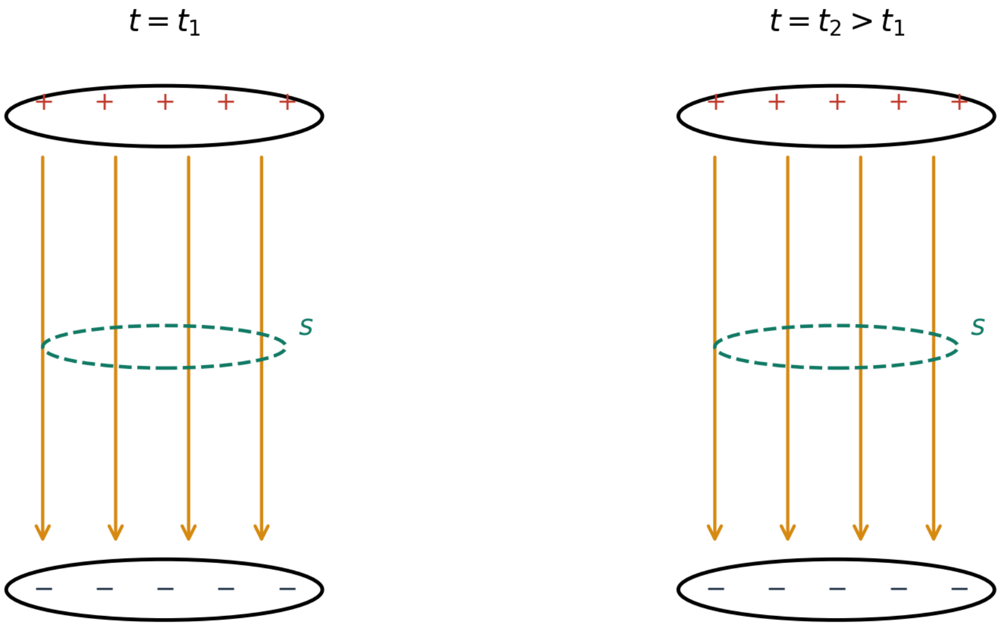

Figure 7. Two snapshots of the charging capacitor at t_1 and $t_2 > t_1$. More charge sits on the plates at t_2; the electric field between them is stronger; the displacement current through any surface threading the gap is proportional to dq/dt = I(t).

Electric field between the plates

Approximating the plates as infinite and ignoring fringing, the field between the plates is uniform with magnitude

$$E(t) = \frac{\sigma(t)}{\varepsilon_0} = \frac{q(t)}{\pi a^2 \varepsilon_0},$$

since the plate area is πa^2 and the surface charge density is $\sigma = q/(\text{plate area})$. The field points from the positive plate to the negative one and is zero outside the gap.

Magnetic field via two Amperian surfaces

Pick a circular Amperian loop of radius s coaxial with the capacitor, lying in the plane between the plates. By symmetry $\mathbf{B} = B(s)\hat{\varphi}$. We treat two cases:

Case (i): inside the gap, s < a

Choose the open surface S to be the flat disk of radius s bounded by the loop. No real current pierces this surface, so $I_{\text{enc}} = 0$, and

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{S} = \mu_0 \varepsilon_0 \frac{dE}{dt} (\pi s^2) .$$

Using $E(t) = q(t) / (\pi a^2 \varepsilon_0)$ and $dq/dt = I(t)$, the left side is $2\pi s B(s)$, giving

$$B(s, t) = \frac{\mu_0 I(t)}{2\pi} \frac{s}{a^2} \qquad (s < a) .$$

Case (ii): outside the plates, s > a

Now the open surface still has area πs^2, but the uniform E only fills the inner disk of radius a. The flux is $\pi a^2 E$, and

$$2\pi s B(s) = \mu_0 \varepsilon_0 \frac{dE}{dt} (\pi a^2) = \mu_0 \frac{dq}{dt} = \mu_0 I(t) ,$$

which gives the familiar wire-style answer:

$$B(s, t) = \frac{\mu_0 I(t)}{2\pi s} \qquad (s > a) .$$

> *Notice the matching at $s = a$: both expressions give $B = \mu_0 I/(2\pi a)$ at the plate edge. The interior solution interpolates linearly from zero on axis up to that boundary value, exactly like the field inside a long straight wire of uniform current density. The displacement current is what makes the math self-consistent across the plate–wire boundary.*

3. Q3: Magnetic Field of Coaxial Cables

Now we move to magnetostatics: a coaxial cable carries current I down the inner conductor and an equal return current up the outer shell. By symmetry the field is azimuthal, $\mathbf{B} = B(s)\hat{\varphi}$, and Ampère's law gives $B(s)$ directly. Two geometries are common on Test 2:

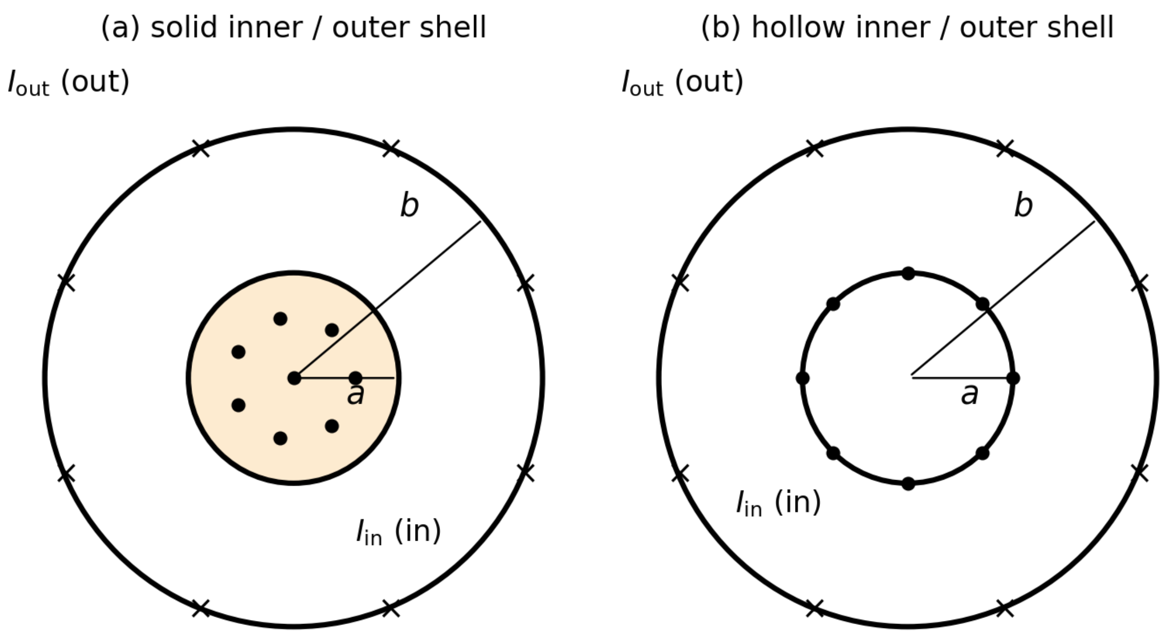

Figure 8. Two coax geometries. Left: solid inner conductor of radius a, outer shell at radius b. Right: hollow inner shell at radius a, outer shell at radius b. Inner current points out of the page (dots); return current is into the page (crosses).

Three regions

Apply Ampère's law on a circular loop of radius s concentric with the axis:

$$\oint \vec{B} \cdot d\vec{\ell} = 2\pi s\, B(s) = \mu_0 I_{\text{enc}}\,.$$

What changes between the two geometries is $I_{\text{enc}}(s)$ in the innermost region.

(a) Solid inner conductor

Assume the current density is uniform inside the solid conductor of radius a: $J = I/(\pi a^2)$. Then I_{enc} grows quadratically inside, saturates between the conductors, and cancels with the return current outside:

$$I_{\text{enc}}(s) = I\,(s/a)^2 \quad (s < a), \qquad I_{\text{enc}}(s) = I \quad (a < s < b), \qquad I_{\text{enc}}(s) = 0 \quad (s > b)\,.$$

Dividing by $2\pi s$ gives the field in each region:

$$B(s) = \frac{\mu_0 I s}{2\pi a^2} \quad (s < a), \qquad B(s) = \frac{\mu_0 I}{2\pi s} \quad (a < s < b), \qquad B(s) = 0 \quad (s > b)\,.$$

Inside the solid conductor the field grows linearly with s; in the gap between conductors it falls as $1/s$; outside the cable the enclosed current is zero, and the field vanishes — this is one of the engineering virtues of a coaxial geometry.

(b) Hollow inner shell

If the inner conductor is instead a thin shell at radius a, then $I_{enc} = 0$ for $s < a$ entirely — no current pierces a loop inside the hollow region:

$$B(s) = 0 \quad (s < a), \qquad B(s) = \frac{\mu_0 I}{2\pi s} \quad (a < s < b), \qquad B(s) = 0 \quad (s > b)\,.$$

The middle region is identical to case (a); only the inner region differs. This is the standard “Faraday cage in reverse” result: a hollow current shell encloses no B field in its interior.

Stored energy and inductance

The energy stored in the magnetic field of an inductor can be obtained two ways. Using the lumped-circuit definition,

$$U = \frac{1}{2} L I^2\,,$$

while the field-theoretic definition uses the magnetic energy density,

$$u(\vec{r}) = \frac{B(\vec{r})^2}{2\mu_0}\,, \qquad U = \int u(\vec{r})\, dV\,.$$

Equating the two expressions gives the inductance per unit length of the coax. We carry out the volume integral explicitly in the next lecture, but the setup is already in place: only the gap region $a < s < b$ contributes (with the correction in case (a) of the inner solid region), and the integral splits into L(length) × $\int s\, ds\, d\varphi$.

> *Big-picture takeaway: in all three problems on this test (multipole expansion, displacement current, coax fields) the strategy is the same. First, identify the symmetry that lets you reduce a vector integral to a scalar one. Second, pick the surface or volume that exploits that symmetry. Third, write the integral form and let it tell you the answer.*